§1.1. Four different kinds of problems appear in the small corpus of known mathematical texts from the Old Akkadian (Sargonic) period, ca. 2340-2200 BC. In texts of the first kind, the area of a rectangle or a nearly rectangular quadrilateral is computed. Examples are ArOr 50, 1 (NBC 7017), MAD 4, 163 (AO 11404), 164 (AO 11405) and 166 (AO 11409) (all in Foster and Robson 2004), as well as DPA 34 and OIP 14, 116 (Ad 786; see Friberg nd, chapter A6.a). In texts of the second kind, here called “metric division exercises,” the short side of a rectangle is computed when the area and the long side of the rectangle are given. To this category belong DPA 38-39, TMH 5, 65, and two privately owned texts, all described in §§2-3 below. In texts of the third kind, “square-side-and-area exercises,” the area of a square is computed when the side is known. To this category belong DPA 36-37, ZA 74, p. 60 (A 5443) and 65 (A 5446) (both in Whiting 1984), and MAD 5, 112 obv. (Ash. 1924.689; Gelb 1970), all described in §§4.2-4.7 below. The only known text of the fourth kind is IM 58045 = 2N-T 600 (Friberg 1990, p. 541), a round hand tablet with a drawing of a partitioned trapezoid with a transversal of unknown length ($§4.8$ below).

§1.2. It has been claimed repeatedly by several authors (Powell, Whiting, and most recently Foster and Robson) that sexagesimal numbers in place value notation must have been used in the complicated computations needed to solve the problems stated in the Old Akkadian metric division exercises and square-side-and-area exercises, always without explicit solution procedures. The aim of the present paper is to show that it is easy to explain those computations in less anachronistic ways. Actually, all known mathematical exercises from the 3rd millennium BC are “metro-mathematical,” in the sense that they are not simply concerned with relations between abstract numbers but rather with relations between measures for lengths, areas, volumes, capacities, etc. In support of this thesis, there is also a brief discussion below of OIP 14, 70, a table of areas of small squares from Adab, ED IIIb ($§4.9$), and of TSS 50 and 671, two metric division exercises from Suruppak of the ED IIIa period ($§4.10$).

§1.3. The conclusion of the discussion in the present paper is that the earliest known firmly dated example of the use of sexagesimal numbers in place value notation is YOS 4, 293 (YBC 1793; see Powell 1976, and $§4.1$ below), a unique “scratch pad” from the Ur III period, ca. 2100-2000 BC, with totals of sexagesimal place value numbers representing traditional weight numbers.

§2. Two privately owned Old Akkadian mathematical texts

§2.1. CMAA 016-C0005, an Old Akkadian metric division exercise of standard type (figure 1)

§2.1.1. The clay tablet from the collection of the California Museum of Ancient Art, Los Angeles (director: J. Berman), shown in figure 1 below, is a small hand tablet inscribed on the obverse with a brief mathematical exercise. The similarity of this text to several previously known Old Akkadian mathematical texts makes it immediately clear that his text, too, is such a text.

§2.1.2. The statement of the problem (the question) is given in a condensed form in lines 1-2:

\[
\begin{align*}
9 \text{ ge}_2 \text{ uš} \\
\text{sag} 1 \text{išu} \text{ aša}_5
\end{align*}
\]

What this means is that a rectangle has a given long side (called uš ‘length’) equal to $9 \text{ ge}_2 = 9 \cdot 60$ (ninda) and a given area (referred to by the determinative [GAN$_2$=] aša$_5$, ‘field’) equal to 1 išu. The obliquely formulated problem is to find the short side (called sag ‘front’). The answer is given in line 3:

\[
2 \text{ kuš}_3 6 \frac{2}{3} \text{ šu-si}
\]

2 cubits $6 \frac{2}{3}$ fingers.
§2.1.3. The length units employed in this text are the ninda (a measuring rod of about 6 meters), the cubit (about 1/2 meter), the finger (about 1.7 cm). The ninda was the main length unit, and the cubit and the finger were fractions of the ninda, with 1 ninda = 12 cubits, 1 cubit = 30 fingers.

Such relations between a series of units can conveniently be expressed in terms of a “factor diagram”:

§2.1.4. Thus, 2 cubits 6 2/3 fingers = 66 2/3 fingers = (66 2/3 ÷ (12 · 30)) ninda = 5/27 ninda, and it follows that the long side of the rectangle is 2,916 times longer than the short side. Such an unrealistic relation between the sides of a field shows that this is an artificially constructed mathematical exercise.

§2.1.5. Being the main length unit, the ninda was often silently understood, as in line 1 of this text. (It was also implicitly understood in many Old Babylonian mathematical texts, and even in certain types of proto-cuneiform texts from the end of the 4th millennium BC). Closely associated with the ninda was the main area unit the iku = 100 square ninda.

§2.1.6. It is, of course, not known how the answer to the question in lines 1-2 of the present text was found. However, a simple and efficient solution algorithm that may have been used starts with a square of area 1 iku. In a number of steps, the square is replaced by a series of progressively longer rectangles of the same area, until in the last step a rectangle is found with the required length and with a front that is the answer to the problem. Here are the successive steps of a factorization algorithm of this kind in the case of CMAA 016-C0005:

1 iku = 10 n. · 10 n.  a square with the side 10 ninda has the area 1 iku
= 1 geš₂ n. · 1 1/2 n. 2 c.  a 6 times longer and 6 times more narrow rectangle of the same area
= 3 geš₂ n. · 1/2 n. 2/3 c.  a 3 times longer and 3 times more narrow rectangle of the same area
= 9 geš₂ n. · 2 c. 6 2/3 f.  a 3 times longer and 3 times more narrow rectangle of the same area

Hence, the answer is that the front is 2 cubits 6 2/3 fingers.

§2.1.7. Indeed, it is easy to check that

1/6 · 10 n. = 1 n. 8 c. = 1 1/2 n. 2 c.,
1/3 · 1 1/2 n. 2 c. = 1/3 n. 2/3 c.,
1/3 · 1/2 n. 2/3 c. = 2 c. 6 2/3 f.

The suggested solution algorithm is of the same kind as some known Old and Late Babylonian factorization algorithms used for similar purposes. For an interesting Late Babylonian example, see Friberg 1999a/2000a.

§2.1.8. It is appropriate to call an exercise of this kind a “metric division exercise”, since the object of the exercise is not to divide a number by another number, but to divide a given area by a given length.

§2.2. ZA 94, 3, an Old Akkadian metric division exercise of a non-standard type (figure 2)

§2.2.1. The Old Akkadian clay tablet shown in figure 2 below is a privately owned medium size hand tablet
inscribed on the obverse with a brief mathematical exercise. It was published in Foster and Robson 2004. The hand copy in figure 2 is based on the copy of the text in ZA 94.

§2.2.2. The text is well preserved, with only a couple of erasures in lines 1-2 and small damaged regions in lines 7-8. Nevertheless, the interpretation of the text is difficult because there are some unusual number notations in lines 1-2, a lost number sign in line 7, and possibly an incorrect or miscopied number sign in line 6. The meaning of the signs in lines 8 and 9 is difficult to establish. In addition, the meaning of the phrase in lines 3-5 is far from clear.

Unfortunately, the present whereabouts of the text are unknown, so the hand copy cannot be collated. It is regrettable that high resolution photographic images of the tablet were not produced at the same time as the hand copy. It is not to be expected that an accurate hand copy can be produced of a difficult text that the copyist does not understand.

§2.2.3. For all these reasons, only conjectural interpretations of the text can be suggested, and it is no wonder that the interpretation proposed below differs in several ways from the interpretation proposed, on rather loose grounds, by Foster and Robson.

§2.2.4. The statement of the problem (the question) is given in a condensed form in lines 1-2:

1. (erasure) 2\(a\) ninda the length
2. sag 2\(di\) \(\text{a}a\) (erasure) \(la_{2}1/4\text{i}kux\) the front: 2/3(iku)? less 1/4(iku)?

§2.2.5. Thus, the question begins in line 1 by stating that the length (of a rectangle) is 2\(a\) ninda. Here, 2\(a\) is a number sign in the form of two horizontal cuneiform wedges (\(a\)), placed one after the other, presumably meaning either 2 or 2 · 60. A quick survey of other Old Akkadian or ED III texts mentioning length numbers has not turned up any parallel number notations for length measures. Normally in such texts, ones are denoted by vertical or slightly inclined cuneiform wedges, and sixties are denoted by slightly larger vertical cuneiform wedges, as for instance in DPA 38-39 (figs. 3, left, and 4), BIN 8, 24 and 147, and in OIP 114, 116 and 163. However, TMH 5, 65 (fig. 3, right) is an exception, since in line 1 of that text both ones and sixties are denoted by curviform (as opposed to cuneiform) horizontal \(a\) signs. Therefore, it is possible that the horizontal number signs in line 1 of ZA 94, 3, are the cuneiform counterparts of the curviform number signs for ones or sixties in TMH 5, 65.

§2.2.6. In line 2, left, the length of the front (of the rectangle) is requested with the single word sag ‘front’, in the same oblique way as in line 2 of CMAA 016-C0005 (fig. 1). Then follows the information that the area of the rectangle is 2/3iku? - 1/4iku?, with 2/3iku? written as 2\(di\) \(\text{a}a\) where 2\(di\) is a number sign in the form of two cuneiform vertical wedges (\(di\)). At least, this is one possible reading of the area number.

Another possibility is to read 2\(di\) \(\text{a}a\) as 2 iku, as it is done in Foster and Robson 2004. However, 2 iku is normally written with two horizontal \(a\)-signs.

§2.2.7. The use of two curviform vertical \(di\)-signs to denote 2/3(?) šar is documented in ED IIIa house purchase contracts such as PBS 9 (1915) no. 3 i 3 (1\(a\a\)c 2\(di\a\)c šar \(e_{2}\)-bi), and TMH 5, 75 obv. i 3 (1\(a\a\)c 2\(di\a\)c šar \(e_{2}\)-bi), both from Nippur. It appears also in the table of small squares OIP 14, 70 (ED IIIb, Adab; fig. 16

Fig. 2. ZA 94, 3. An Old Akkadian metric division exercise of a non-standard type.
below), where the square of 10 cubits is expressed as follows:

\[
\begin{align*}
10 \text{ kuš}_3 & \quad \text{sa}_2 \\
2 \text{ dišar} & \quad \text{šar} \quad 2 \text{ dišar} & \quad \text{gin}_2 \\
\text{la}_2 & \quad 1 \text{ sa}_10 & \quad \text{ma-na}
\end{align*}
\]

What this means is that

\[
10 \text{ cubits squared} = 2/3 \text{ šar} \quad 2 \text{ shekels} - 1 \text{ exchange-mina},
\]

where

\[
1 \text{ exchange-mina} = 1/3 \text{ shekel}.
\]

§2.2.8. This equation can be explained as follows (cf. Friberg nd, app. 1, under fig. A1.4):

1 reed = 1/2 ninda
1 sq. reed = 1/4 sq. ninda = 15 (area-)shekels
1 sq. cubit = 1/36 sq. reed = 1/36 \cdot 15 shekels
= 1/12 \cdot 5 shekels = 1/12 \cdot 15 exchange-minas
= 1/12 \cdot 1/4 \cdot 15 exchange-minas
= 1 1/4 exchange-mina

\[
\text{sq. (10 cubits)} = 100 \cdot 1 1/4 \text{ exchange-mina}.
\]

\[
= 125 \text{ exchange-minas} = 1/2 1/4 \text{ shekel},
\]
or simply

\[
1 \text{ exchange-mina} = 1/3 \text{ shekel}.
\]

§2.2.9. Notations for fractions resembling some of those used in the ED IIIb Agade text OIP 14, 70, can be found also in BIN 8, 175 (=NBC 6915; Edzard 1968, no. 54), an Old Akkadian slave sale contract from Nippur. There the sum of 4 shekels of silver (4 kuš gin2), 1/2 shekel (1/2 kuš gin2), 1/4 shekel (kuš igi 4-gal-kam), 4 shekels, 2 shekels, and 2 exchange-minas (kuš sa10+2aš ma-na-kam) is given as 11 1/2 shekels - 15 exchange-shekels, written as 11 1/2 la2 sa10+15 kuš gin2. The summation shows that

\[
10 1/2 1/4 \text{ shekels} + 2 \text{ exchange-minas} = 11 1/2 \text{ shekels} - 15 \text{ exchange-shekels}.
\]

This equation can be reduced to

\[
2 1/4 \text{ exchange-minas} = 1/2 1/4 \text{ shekel},
\]
or simply

\[
1 \text{ exchange-mina} = 1/3 \text{ shekel}.
\]

§2.2.10. It is interesting to note that a mix of curviform and cuneiform number signs are used in OIP 14, 70 (fig. 16 below). Thus, curviform horizontal aš-signs are used in front of šar (square ninda), kuš (cubit), gi (reed = 1/2 ninda), and sa10 ma-na (exchange-mina), while cuneiform, slanting diš-signs are used in front of gin2 (shekel). For the basic fractions of the šar, both curviform and cuneiform number signs are used in OIP 14, 70, with curviform signs for 1/2 and 2/3, but a cuneiform sign for 1/3. The curviform and cuneiform signs, respectively, for the three basic fractions are:

\[
\begin{align*}
\text{šanabi} (1\frac{2}{3}, 2\text{ dišar} \text{šar}) & = \text{maš} (1\frac{1}{2}, 1\text{ dišar} \text{šar}) \\
\text{išanān} (1\frac{1}{3}, 1\text{ dišar} \text{šar}) & = 1\frac{1}{3} \text{ shekel}
\end{align*}
\]

These basic fractions could also be used for fractions of the 60-shekel mina. See, for instance, OIP 14, 76, where the curviform signs are used for both 2/3 and 1/3, as fractions of a mina.

§2.2.11. So far, the cuneiform form above of the sign for 2/3 seems to be documented only, possibly, in ZA 94, 3. For the sake of completeness, it must be remarked here that there were also other ways of writing 2/3 in ED IIIb texts. The following two examples are taken from OIP 14, 48 and 49:

\[
\begin{align*}
\text{LI} & \quad \text{LI} (\text{followed by the word ša-na-pi}) \\
\text{LI} & \quad \text{LI}
\end{align*}
\]

Note that in the Old Akkadian mathematical texts in figs. 9, 11, and 13 below, the sign for 2/3 is different again and is followed by -ša

§2.2.12. The meaning of the last number sign in line 2 of ZA 94, 3 is possibly 1/4 iku. See Powell, ZA 62, 217-220. According to Powell, the following notations for iku fractions, otherwise unknown, occur in BIN 8, nos. 49, 51, 112, 114, 120 (ED IIIb), and 189, 190, 195, 199 (Old Akkadian):
§2.2.13. The answer to the question in ZA 94, 3 is given in lines 6-7 of the text:

sag-bi 4⅔ kuš₃ numun
1 šu-du₃-a [5]šu-si

Its front (is) 4 seed-cubits

The relative sizes of the length measures mentioned here are given by the following factor-diagram:

\[
\begin{array}{c}
1 \text{ iku} \\
\downarrow 2 \quad \downarrow 2 \quad \downarrow 2 \\
\text{½ iku} \quad \text{¼ iku} \quad \text{⅛ iku}
\end{array}
\]

\[A_{\text{var}}: \text{ }}

§2.2.14. The text in lines 3-5 of ZA 94, 3 is, conceivably, part of either the solution algorithm or the answer. Its meaning is far from clear. Here is a partial translation of the obscure passage:

3. ša₃-ba šu-du₃-ru²
   Inside it X.

4. igi 6-gal₂-bi b₁₂-gar
   Its 6th-part set,

igi 4-gal₂-bi b₁₂-gar
   its 4th-part set.

5. ba-pa
   It is found.

§2.2.15. If everything with the text had been in order, it should now be possible to show that in ZA 94, 3 the given area is equal to the product of the given length and the computed front. However, this is easier said than done. Apparently, the one who wrote the text made a couple of mistakes. It is also possible that the hand copy of the text in ZA 94, 3, contains one or several mistakes, for instance a mistake in line 6, where what looks like a 4 in front of a small damaged kuš₃ sign may, conceivably, be a 1 in front of a larger kuš₃ sign.

§2.2.16. In any case, here follows a proposed reconstruction of the successive steps of the solution algorithm in ZA 94, 3, based on the assumption that the given area A is 2/3 iku - 1/4 iku, as suggested above, and on the further assumption that the given length u is 2(· 100)³, rather than 2(· 60)² <ninda>.

1. \(A/200 = (2/3 \text{ iku} - 1/4 \text{ iku})/200 = (2/3 \text{ šar} - 1/4 \text{ šar})/2 = (40 - 15) \text{ area-shekels}/2 = 12 1/2 \text{ area-shekels.}

2. \(12 1/2 \text{ area-shekels} = 1/6 \text{ area-šar} + 1/4 \text{ of } 1/6 \text{ area-šar.}

3. \((1/6 \text{ area-šar} + 1/4 \text{ of } 1/6 \text{ area-šar})/1 \text{ ninda} = 1/6 \text{ ninda} + 1/4 \text{ of } 1/6 \text{ ninda} = 1 1/4 \text{ seed-cubit.}

4. Hence, the front s = \(A/u = 1 1/4 \text{ seed-cubit} 1 \text{ šu-du₃-a} 5 \text{ fingers.}

§2.2.17. It is likely that the detour over the composite fraction \(1/6 + 1/4 \cdot 1/6\) was necessary for the reason that it would not have been legitimate to speak about ‘12 1/2 shekels of a ninda’. Note that the use of a composite fraction of a similar type is documented in the Old Babylonian combined work norm exercise MCT 81 (YBC 7164) §7 (Friberg 2000b, 127). There, a man works ‘1/5 of a day throwing’ (mud) and ‘2/3 of a day and 1/5 of 2/3 of a day basketing’ (mud). (It is easy to check that \(2/3 + 1/5 \cdot 2/3 = 4/5\), since \(5 \cdot (2/3 + 1/5 \cdot 2/3) = 6 \cdot 2/3 = 4\).

§2.2.18. The proposed explanation of the solution algorithm in ZA 94, 3, is simultaneously a possible explanation of the obscure passage in lines 3-5 of ZA 94, 3, where the phrase ‘its 6th-part set, its 4th-part set’ may mean ‘take the 6th-part of 1 ninda, and (add) the 4th-part of that’.

§2.2.19. It is awkward that the proposed explanation of ZA 94, 3, hinges on the assumption that \(2\mathbb{a}_5\) in line 1 of ZA 94, 3, means 2 · 100 rather than 2 · 60. It is also awkward that the area number recorded in line 2 of ZA 94, 3, suffers from what may be a fatal flaw, namely the assumption that \(2\mathbb{a}_5\) can stand for \(2/3\) iku. The fractions of the iku that are known to occur in other cuneiform texts are \(1/2\), \(1/4\), and \(1/8\) iku. In addition, the only known examples of the occurrence of \(2\mathbb{a}_5\) as a sign for \(2/3\) are the ones mentioned above, where in all cases \(2\mathbb{a}_5\) šar stands for \(2/3\) šar. Therefore, maybe it would be wise to follow Foster and Robson in assuming that the area number given in line 2 of ZA 94, 3, is meant to be a fraction of the šar rather than of the iku.

In that case, the author of this text, probably a student who listened to his teacher’s oral instructions, made the double mistake of writing \(\mathbb{a}_5\) instead of šar and of using a sign for \(1/4\) iku as a sign for \(1/4\) šar.

§2.2.20. Thus, a slightly different alternative interpretation of ZA 94, 3, is that the length number \(2\mathbb{a}_5\) in line 1 of the text simply stands for \(2 \mathbb{a}_5\), not 200’ <ninda>, and that the curiously written area number in line 2 stands for \(2/3\) šar¹ - 1/4 šar¹, not \(2/3\) iku - 1/4 iku. In that case, the first step of the solution algorithm proposed above can be changed to the more straightforward

\[1. \ A/2 = (2/3 \text{ šar} - 1/4 \text{ šar})/2 = (40 - 15) \text{ area-shekels}/2 = 12 1/2 \text{ area-shekels.}

The remaining three steps of the proposed solution algorithm are not affected by the change.
§3. Three earlier published Old Akkadian metric division exercises

§3.1. In ZA 94, 3, Foster and Robson briefly mention three earlier published Old Akkadian metric division exercises and claim that in all three of them the answer was found by use of sexagesimal numbers in place value notation. This is almost certainly not correct. The alternative discussion below of the three metric division exercises in question is borrowed (with some amendments) from Friberg nd, app. 6, A6 c. The hand copies in figures 3-4 are based on copies of the texts in DPA and TMH 5.

§3.2. DPA 38 and TMH 5, 65, Old Akkadian metric division exercises of standard type (figure 3)

Note that DPA 38 (=PUL 29) is erroneously called PUL 54 in ZA 94, 3.

§3.2.1. This is the brief text of DPA 38 (according to Limet probably from Girsu/Lagash):

2ge₂ uš 2 4u u₂
sag 1iku ašš₃
sag-bi 3 ku₅₃ numun
1 GEŠ.BAD 1 ŠU.BAD

§3.2.2. In order to understand this and the following two exercises, it is necessary to be familiar with the following nearly complete version of the factor diagram for Old Akkadian ninda fractions:

\[ \text{L(OAkk)} : \quad \begin{array}{cccc}
\text{ninda} & \text{ninda} & \text{ku₅₃} & \text{numun} \\
\text{GEŠ.BAD/ku₅₃} & \text{GEŠ.BAD/ku₅₃} & \text{GEŠ.BAD/ku₅₃} & \text{GEŠ.BAD/ku₅₃}
\end{array} \]

\[ \quad \begin{array}{cccc}
\text{ŞU.BAD} & \text{şu-du₃-a} & \text{şu-si} & \text{şu-si}
\end{array} \]

§3.2.3. In DPA 38 the front \( s \) (the short side of a rectangle) has to be computed when the length \( u = 2 \text{ ge}_₂ 40 \) (ninda) and the area \( A = 1 \text{ iku} \). The answer is given in the form

\[ s = 3 \text{ seed-cubits} 1 \text{ GEŠ.BAD} \]

\[ 1 \text{ ŞU.BAD} \]

(= \( 3 \frac{1}{2} \frac{1}{4} \) seed-cubits).

§3.2.4. It is clear that here 2 ge₂ 40 (just like 9 ge₂ in CMAA 016-C0005) is a regular sexagesimal number, since 2 ge₂ 40 = 160 = 32 \cdot 5 = 2^5 \cdot 5. However, before the invention of place value notation and abstract numbers the front cannot have been computed as (the number for) the area times the reciprocal of (the number for) the length. Instead, the front may have been computed by use of a factorization algorithm like the one proposed above in the case of CMAA 016-C0005. In the case of DPA 38, the factorization algorithm can have taken the following form:

\[ 1 \text{ iku} = 10 \text{ n.} \cdot 10 \text{ n.} \quad \text{a square with the side 10 ninda has the area 1 iku} \]

\[ = 40 \text{ n.} \cdot 2 \text{ n.} 3 \text{ s.c.} \quad \text{one side multiplied by 4, the other by } 1/4 \]

\[ = 2 \text{ ge}_₂ 40 \text{ n.} \cdot 3 \frac{1}{2} 1/4 \text{ s.c.} \quad \text{the length multiplied by 4, the front by } 1/4 \]

Hence, the answer is that the front is \( 3 \frac{1}{2} \frac{1}{4} \text{ s.c.} = 3 \) seed-cubits 1 GEŠ.BAD 1 ŞU.BAD.

§3.3.1. TMH 5, 65 (Westenholz 1975, no. 65) is a similar text, probably from Nippur. The length \( u = 1 \text{ ge}_₂ 7 \frac{1}{2} \) ninda, the area \( A \) is again \( 1 \text{ iku} \), and the answer is given in the form

\[ s = 1 \text{ ninda} \text{nindax(DU)} 5 \text{ ku}_₃ \]

\[ (= 1 \text{ ninda} 5 \frac{2}{3} \frac{2}{3} \text{ a-şu-du₃-a 3 şu-si 1/3 şu-si} \text{ cubits } 3 \frac{1}{3} \text{ fingers}) \]

§3.3.2. The ninda fractions appearing in this text are not quite the same as those in DPA 38. However, the given length is here, again, a regular sexagesimal number (times 1 ninda), since 1 ge₂ 7 \( \frac{1}{2} \frac{1}{2} \) = 27 \cdot 2 \( \frac{1}{2} \frac{1}{2} \) = 27/4 \cdot 10. The solution can be obtained by use of the following factorization algorithm:

\[ 1 \text{ iku} = 10 \text{ n.} \cdot 10 \text{ n.} \quad \text{a square with the side 10 ninda has the area 1 iku} \]

\[ \quad \text{the front is } 3 \frac{1}{2} \frac{1}{4} \text{ s.c.} = 3 \text{ seed-cubits} 1 \text{ GEŠ.BAD} 1 \text{ ŞU.BAD}. \]

Fig. 3. DPA 38 and TMH 5, 65, two Old Akkadian metric division exercises.
Fig. 4. DPA 39, an Old Akkadian mathematical assignment with a metric division problem.

Thus the answer in this case would be

\[ s = 4 \ 2/3 \text{ cubits} \ 8 \text{ fingers} \ 2/3 \text{ barley-corn} \]

\[ \text{and} \ \frac{1}{3} \text{ of} \ 2/3 \text{ barley-corn}. \]

§3.3.6. However, this answer would not have fitted into the small space available on the clay tablet! That may be the reason why no answer is recorded there.

§3.3.7. It is remarkable that the three length numbers

\[ 2 \text{ geš}_2 \ 40 \ n. = (1 + 1/3) \cdot 2 \text{ geš}_2, \]

\[ 1 \text{ geš}_2 \ 7 \ 1/2 \ n. = (1 + 1/8) \cdot 1 \text{ geš}_2, \]

\[ \text{and} \ 4 \text{ geš}_2 \ 3 \ n. = (1 + 1/90) \cdot 4 \text{ geš}_2, \]

in \textit{DPA 38}, \textit{TMH 5, 65}, and \textit{DPA 39} all are regular sexagesimal numbers (times 1 ninda). These are the earliest known attestations of (deliberately chosen) regular sexagesimal numbers, foreshadowing the enormous interest attached to such numbers in both Old and Late Babylonian mathematical texts. This point was missed by Foster and Robson, who even suggested (2004, footnote 9) that "perhaps 4 03 is a scribal error for 4 30."

§4. On true and alleged antecedents of Old Babylonian sexagesimal place value notation

§4.1. Sexagesimal place value numbers in texts from the Neo-Sumerian Ur III period

§4.1.1. M. Powell was the first person to seriously study mathematical cuneiform texts from the 3rd millennium BC. In \textit{Historia Mathematica} 3 (1976) pp. 417-439,
he wrote about “The antecedents of Old Babylonian place notation and the early history of Babylonian mathematics.” He correctly noted that sexagesimal place value notation was used in YOS 4, 293, a (non-mathematical) Ur III text, which can be dated, on the basis of a year formula (‘the year Enunugalanna was installed as en-priest of Inanna in Uruk’) to the fifth year of Amar-Suen, 2043 BC. Therefore, it seems to be definitely established that place value notation for sexagesimal numbers was already in use before the end of the 3rd Dynasty of Ur.

§4.1.2. Powell commented the physical appearance of YOS 4, 293 (see fig. 5) in the following way:

But this document has a greater significance than is evident from the cuneiform copy. When I collated this text in the Yale Babylonian Collection (June 17, 1974), it turned out to be a kind of ancient ‘scratch pad’. It has a form similar to a school text, being rather thick and having flat edges. The writing surface is extremely flat, and the back side, which is not used, is convex. The writing surface shows clear traces of having been previously used. The appearance of the tablet suggests that it was moistened and smoothed off after use. Here we have at last an explanation of why so little trace of sexagesimal notation has survived from the Ur III period. [Here Powell adds a footnote mentioning other examples he knows about.] Calculations in sexagesimal notation were made on temporary tablets which were then moistened and erased for reuse after the calculation had been transferred to an archival document in standard notation.

§4.1.3. Two columns of sexagesimal numbers in place value notation are recorded on the obverse of YOS 4, 293, one (here called A) neatly in the first text case, the other one (here called B), in some disorder after the apparent end of the inscription. The numbers in columns A and B are the following:

\[
\begin{array}{c|c}
A & B \\
\hline
14 54 & 2 54 \\
29 56 50 & 45 \\
17 43 40 & 28 \\
30 53 20 & 17 \\
& 2 28 \\
& 27 \\
\end{array}
\]

The sums of the numbers in the two columns are not recorded, but clearly they are, respectively,

\[
\begin{align*}
\text{sum } A &= 1 33 27 40, \\
\text{sum } B &= 7 19.
\end{align*}
\]

These two totals have to be interpreted as

\[
\begin{align*}
\text{sum } A &= 1 33;27 40 \text{ shekels, and} \\
\text{sum } B &= 7 19 \text{ shekels.}
\end{align*}
\]

§4.1.4. The totals can be converted to ordinary Sumerian/Babylonian weight numbers as follows:

\[
\begin{align*}
1 33;27 40 \text{ shekels} &= 1 \ 1/2 \text{ mina} 3 \ 1/2 \text{ shekel} - 7 \text{ barley-corns} (1 \text{ barley-corn} = 1/180 \text{ shekel} = ;00 20 \text{ shekel}) \\
&= 1 1/2 \text{ mina} 3 1/2 \text{ shekel} - 7 \text{ barley-corns}
\end{align*}
\]

and

\[
7 19 \text{ shekels} = 7 \text{ minas} 19 \text{ shekels}.
\]

§4.1.5. As shown in the “conform transliteration” of YOS 4, 293, in figure 5, these totals are actually recorded on the clay tablet. Sum A is specified as muku(DU) didli, probably meaning ‘diverse deliveries’,
and sum B as mu-kumu(DU) a-tu5-a lugal ‘deliveries for the ritual cleaning of the king’. In addition, the grand total is also recorded (twice), that is the sum of these two totals:

\[
\text{sum (A + B)} = \text{sum A} + \text{sum B} \\
\text{= 1 } 33;27 \text{ 40 shekels } + 7 \text{ 19 shekels} \\
\text{= 8 } 52;27 \text{ 40 shekels} \\
\text{= 8 } 5/6 \text{ minas } 2 \text{ 1/2 shekels} \\
\text{- 7 barely-corns.}
\]

§4.1.6. The first grand total is specified as ša₃ im-u₄ ‘from the daily tablets’, the second grand total (possibly) as ša₃ [bala]-a 'obligatory payments'.

§4.1.7. The numbers in column A are the place value notation equivalents of the following numbers:

| A     | 14 54 shekels = 14 5/6 minas 4 shekels,  |
|       | 29 56:50 shekels = 29 5/6 minas 5 1/6 shekels, |
|       | 17 43:40 shekels = 17 2/3 minas 3 2/3 shekels, |
|       | 30 53:20 shekels = 30 5/6 minas 3 1/3 shekels. |

Similarly, the numbers in column B can be written in standard notation for weight numbers as

| B     | 2 54 shekels = 2 5/6 minas 4 shekels,  |
|       | 45 shekels = 2 1/3 mina 5 shekels,  |
|       | 28 shekels = 1/3 mina 8 shekels,  |
|       | 17 shekels = 17 shekels,  |
|       | 2 28 shekels = 2 1/3 minas 8 shekels,  |
|       | 27 shekels = 1/3 mina 7 shekels.  |

§4.1.8. In YOS 4, 293, there are two ways of writing the digits 4, 7, 8, 9, and 40, either in the "traditional", Sumerian way with horizontally extended number signs, or in the "new" way (the one most often used in Old Babylonian cuneiform texts) with vertically extended signs. In some Old Babylonian mathematical texts, such as the famous table text MCT 38 (Plimpton 322), the new way of writing digits is used for place value numbers in tables and computations, while the traditional way of writing digits is used in ordinary numbers, such as line numbers or traditional metrological numbers. The same tendency is clear in YOS 4, 293, where the traditional way of writing digits is used in the totals and grand totals, while the new way is used, although not quite consistently, in the place value numbers. It is likely that the more compact vertical form was invented in order to make it easier to fit the number signs of long sexagesimal numbers into narrow columns, optimally with ones and tens in orderly columns as in the first text case of YOS 4, 293. (Surprisingly, both ways of writing 4 are used in line 1 of YOS 4, 293, and both ways of writing 40 in line 3.) Note in this connection Powell’s remark (1976, 421)

Moreover, another unusual characteristic of this text suggests how the Sumerian system of (place value) notation functioned without a sign for zero: the sexagesimal numerals are arranged (in A and B) in quite clear columns according to their proper power.

§4.1.9. So far, the only known example of a mathematical cuneiform text from the Ur III period using sexagesimal numbers in place value notation is the small brick metrology exercise RTC 413 (see Friberg, 1990, fig. 3).

§4.1.10. It is difficult to imagine that sexagesimal numbers in place value notation could be widely used without recourse to sexagesimal multiplication tables and tables of reciprocals. Although arithmetic table texts inscribed almost exclusively with numbers are hard to date, there are several reasons to believe that a small group of distinctly atypical tables of reciprocals are from Ur III. The group in question includes the Nippur text MKT 1, 10 (HS 201; s. Oelsner, ChV [2001]), the Nippur text Ist. Ni. 374 (Proust 2004,
ch. 6.2.2 and Table 1), and the Tello/Girsu text ITT 4, 7535 (s. Friberg nd, app. 1, fig. A1.2).

According to Proust 2004, 118, four more texts of the same type have been found by B. Lafont in Istanbul, and two more, both from Umma, have been found by E. Robson at the British Museum.

§4.1.11. The conform transliteration of Ist. Ni. 374 in figure 6 is based on a hand copy in Proust 2004. Note the similarity with YOS 4, 293: both tablets are of the same form and are inscribed in two columns. The traditional forms of the digits 4, 7, 8, 9, and 40 are used everywhere in Ist. Ni. 374. However, also here there seems to be a certain difference between the forms of the digits used for the ordinary numbers n and those used for the sexagesimal place value numbers recorded in the form igi n. In the former, the digits 40 and (probably) [50] are written with one line of tens, while in the latter, 40 and 50 are written more compactly with two lines of tens.

§4.2. The alleged use of sexagesimal place value numbers in Old Akkadian texts.

§4.2.1. After discussing YOS 4, 293, Powell went on to the Old Akkadian mathematical text DPA 38 (see above, fig. 3, left), suggesting that the metric division problem in that text was solved as follows:

\[ s = \frac{A}{u} = \frac{1\ 40/\ 2\ 40}{1\ 40/\ 2\ 40} = 1\ 40 \cdot \frac{1}{2\ 40} = 1\ 40 \cdot 0;00\ 22\ 30 = 0;37\ 30, \]

where

\[ s = \frac{A}{u} = 1\ 40\ sq.\ ninda/\ 2\ 40\ ninda = 0;37\ 30\ ninda = 3\ seed-cubits\ 1\ cubit\ 1\ half-cubit. \]

§4.2.2. In the same vein, he suggested that another metric division problem in the Old Akkadian mathematical text DPA 39 (above, fig. 4) was solved as follows:

\[ s = \frac{A}{u} = \frac{1\ 40/\ 4\ 03}{1\ 40/\ 4\ 03} = 1\ 40 \cdot \frac{1}{4\ 03} = 1\ 40 \cdot 0;00\ 14\ 48\ 53\ 20 = 0;24\ 41\ 28\ 53\ 20 (ninda) = ? \]

§4.2.3. Also the computation producing the answer in TMH 5, 65 (above, fig. 3, right) was explained by Powell in a similar way, namely as

\[ s = \frac{A}{u} = \frac{1\ 40/\ 1\ 07;30}{1\ 40/\ 1\ 07;30} = 1\ 40 \cdot \frac{1}{1\ 07;30} = 1\ 40 \cdot 0;00\ 53\ 20 = 1;28\ 53\ 20 (ninda) = 1\ n.\ 5^{2/3}\ c.\ 3^{1/3}\ f. \]

§4.2.4. Powell’s suggested explanations were repeated, without further discussion of the matter, in three “summaries of calculation” in Foster and Robson, ZA 94, 3. However, as was shown above, the metric division problems in the three mentioned Old Akkadian texts can easily have been solved by use of simple factorization algorithms, and there is no evidence for calculations with sexagesimal place value numbers in these or any other Old Akkadian mathematical texts.

§4.2.5. Also following in Powell’s footsteps, Whiting wrote in ZA 74 (1984) pp. 59-66 a paper with the title “More Evidence for Sexagesimal Calculations in the Third Millennium B.C.” In that paper, he discussed the three Old Akkadian square-side-and-area exercises DPA 36-37, and ZA 74, p. 60 (A 5443). He thought he could find evidence for counting with sexagesimal numbers in place value notation in all three of them. However, a close look at Whiting’s arguments will show that they are based on an inadequate understanding of the meaning of place value notation, and on a lacking familiarity with the peculiarities of pre-Babylonian mathematical cuneiform texts (cf. the detailed discussion of mathematical cuneiform texts from the 3rd millennium BC in Friberg, nd, ch. 6 and app. 6). The three square-side-and-area exercises mentioned by Whiting will be discussed again below. The vector graphic copies of the author in figures 7, 9, and 11 are based on the copies of the texts in Limet 1973 and Whiting 1984.

§4.3. DPA 36, an Old Akkadian square-side-and-area exercise (figure 7)

§4.3.1. Take, for instance, DPA 36. In that text, the area of a square with the side

\[ 11\ ninda\ ninda_3 = 1\ ku_3 \]

was found

\[ a-bi = 1/ku_1/4/ku\ aša_4 \]

\[ 2\ 1/2\ šar\ 6\ a\ gin_2 \]

\[ 15\ gin_2\ tur \]

\[ ba-pa\ (is\ found) \]

\[ 3\ cm \]

This is an early example of a “funny number”. See Friberg 1990, §5.3 a)

Appearing in this text are the following fractions of a šar:

\[ 1\ gin_2 = \frac{1}{60}\ šar \]

\[ 1\ gin_2\ tur = \frac{1}{60}\ gin_2 \]

Fig. 7. DPA 36, an Old Akkadian square-side-and-area exercise.
§4.3.2. Whiting expresses both the side and the area of the square in DPA 36 in terms of sexagesimal numbers in place value notation, stating that

The side of the square is given as 11:17, 30 ninda and the correct area is 2,7,30,6,15 sar. The answer given in the tablet is 1 1/4 (iku) GANA 2 2 1/2 sar 6 gin 2 15 gin 2-tur which Powell interprets as 2,7,36,15. However, the answer given by the student need not be considered an error if we allow the positional use of the gin 2 sign. In this case we would interpret 1/2 sar 6 gin 2 15 gin 2-tur as ;30,6,15 which is the correct answer. Note that this is the same problem that would arise if sexagesimal place notation were used since 30,6 and 36 are both written with exactly the same characters in this system.

§4.3.3. Thus, both Powell and Whiting assume that the area of the square was computed by first converting the given length number to a sexagesimal number in place value notation, then squaring this number, and finally converting the resulting sexagesimal number back to an area number. In the process, a simple positional error resulted in writing 6 gin 2 15 gin 2 tur (= 6 1/4 gin 2 tur) instead of 6 1/4 gin 2 tur.

§4.3.4. Contrary to the mentioned assumptions by Powell and Whiting, there is really no need to postulate that the author of DPA 36 counted with sexagesimal numbers in place value notation. Instead, he can have started by observing that since the ku₃ numun 'seed-cubit' = 1/6 ninda, and since the GEŠ.BAD = 1/2 seed-cubit and the SU.BAD = 1/4 seed-cubit, the given length of the side of the square can be interpreted as a "fractionally and marginally expanded length number". Indeed, it can be expressed in the following form:

\[ s = 11 \text{ninda} 1 \text{ku₃ numun} 1 \text{GEŠ.BAD} 1 \text{SU.BAD} \]

is given as

\[ A = 1 \text{iku} \frac{1}{4} \text{iku} aš₃ 2 \frac{1}{2} šar 6 \text{gin} \frac{1}{2} \text{gin} \text{tur}. \]

\[ \text{sq.} (1 \text{SU.BAD}) = 6 \frac{1}{4} \text{gin} \text{tur} \]

\[ \text{SU.BAD} = \frac{1}{24} \text{ninda} = \frac{1}{240} \text{of} 10 \text{ninda}. \]

§4.3.5. The (area of the) square of this length number can have been be computed in a few simple steps, with the first step reasonably being to compute the square of only the first two terms:

\[ \text{sq.} s = \text{sq.} \left( 10 \text{ninda} + \frac{1}{8} \cdot 10 \text{ninda} \right) \]

\[ = \text{sq.} \left( 10 \text{ninda} + \frac{1}{8} \cdot 10 \text{ninda} \right) + 2 \cdot \left( 10 \text{ninda} + \frac{1}{8} \cdot 10 \text{ninda} \right) \cdot \frac{1}{4} \text{seed-cubit} \]

\[ = 1 \frac{1}{4} \text{iku} \frac{1}{2} \text{šar} 3 \frac{1}{2} \text{gin} \frac{1}{2} \text{gin} \text{tur} + \frac{1}{2} \left( 25 \text{gin} \frac{1}{2} \cdot \frac{1}{8} \cdot 25 \text{gin} \text{tur} ight) + \frac{1}{6} \frac{1}{2} \text{gin} \text{tur} \]

\[ = 1 \frac{1}{4} \text{iku} \frac{1}{2} \text{šar} 6 \frac{1}{4} \text{gin} \text{tur}. \]

Geometrically, a computation like this can be explained as an application of the easily observed "square expansion rule" that the square of a length composed of two unequal parts is a square composed of four parts, namely two unequal squares and two equal rectangles. See figure 8.

§4.3.6. In the second step of the computation, a second application of the same rule shows that

\[ \text{sq.} (1 \text{SU.BAD}) = \frac{1}{4} \text{seed-cubit} = \frac{1}{4} \cdot \frac{1}{6} \text{ninda}, \text{it follows that} \]

\[ 1 \text{ninda} \cdot 1 \text{SU.BAD} = \frac{1}{4} \cdot \frac{1}{6} \text{sq. ninda} \]

\[ = \frac{1}{4} \cdot \frac{1}{6} \text{šar} = \frac{1}{4} \cdot 10 \text{gin} \]

\[ = 2 \frac{1}{2} \text{gin} \]

\[ \text{sq.} (1 \text{SU.BAD}) = \frac{1}{4} \cdot \frac{1}{6} \cdot 2 \frac{1}{2} \text{gin} \text{tur} \]
§4.3.7. The calculation of sq. s above was based on these relations. It is likely that they were supposed to be well known by Old Akkadian school boys, but that the author of DPA 36 did not correctly remember the value of sq. (1 \( \sum U \), BAD), which he took to be 6 1/4 gin\(^2\) tur instead of 6 1/4 gin\(^2\) tur.

§4.4. DPA 37, another Old Akkadian square-side-and-area exercise (figure 9)

§4.4.1. Now consider the parallel text DPA 37, in which the side and the area of a square are

\[
s = 1 \frac{1}{2} ar^2 ninda - 1 \text{ seed-cubit},
\]

and

\[
A = 2 \frac{1}{2} ar^2 49 \text{ bur}^3 5 \text{ iku} 1/8 \text{ iku} \\
5 1/2 \frac{1}{3} \frac{1}{2} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1/2}
§4.4.5. This computation can be explained by use of the factor diagrams below for sexagesimal numbers and large area measures, respectively.

These are the cuneiform variants of the number signs, as opposed to the round, curviform variants.

§4.4.6. In the second step of the computation in DPA 37, an application of the square contraction rule shows that

\[
\text{sq.} (1 \text{ \ar2-gal 5 \ge2 n. } - 1 \text{ s.c.}) = \text{sq.} (1 \text{ \ar2-gal 5 \ge2 ninda}) - 2 \cdot (1 \text{ \e3 1/2 iku}) + \text{sq.} 1 \text{ s.c.}
\]

This computation can be explained by use of the following multiplication table for length measures:

- 1 s.c. · 1 s.c. = \( 1/6 \cdot 1/6 \text{ \ar} \) (s.c. = seed-cubit)
- 1 n. · 1 s.c. = \( 1/6 \cdot 10 \text{ \gin} \)
- 10 \text{ \gin} ,
- 1 \text{ \ge2 n. · 1 s.c.} = \( 1/6 \text{ \ar} 
- 10 \text{ \gin} ,
- 1 \text{ \ar2 n. · 1 s.c.} = \( 1/6 \text{ \ar} 
- 10 \text{ \ge2} ,
- 1 \text{ \ar2 n. · 1 s.c.} = \( 1/6 \text{ \ar} 
- 10 \text{ \ar} ,
- 1 \text{ \ar2 n. · 1 s.c.} = \( 1/6 \text{ \ar} 
- 1 \text{ \e3}.

§4.4.7. In view of the computations above, the correct answer in the case of DPA 37 ought to be

\[
\text{sq.} (1 \text{ \ar2-gal 5 \ge2 n. } - 1 \text{ s.c.}) = 2 \text{ \ar2-gal 20 \ar2 50 \bur3} - 2 \cdot (1 \text{ \ge2 1/2 iku}) + 1/6 \cdot 1/6 \text{ \ar}.
\]

However, the answer given in the text is slightly different, namely

\[
\text{sq.} (1 \text{ \ar2-gal 5 \ge2 n. } - 1 \text{ s.c.}) = 2 \text{ \ar2-gal 20 \ar2 49 \bur3} 5 \text{ 1/8 iku}
\]

The difference between the correct and the recorded answer is

\[
1/8 \text{ iku 5 1/2 } = 12 1/2 \text{ \ar} + 5 1/2 \text{ \ar} = 18 \text{ \ar}.
\]

§4.4.8. This curious error can possibly be explained as follows: The student who wrote the text may have become confused by the \( 1 \text{ \e3 1/2 iku} \) resulting from an intermediate step of the computation, and counted in the following incorrect way:

\[
\text{sq.} (1 \text{ \ar2-gal 5 \ge2 n. } - 1 \text{ s.c.}) = 2 \text{ \ar2-gal 20 \ar2 50 \bur3} - 2 \cdot (1 \text{ \e3 1/2 iku}) + 1/6 \cdot 1/6 \cdot \text{sq.} \text{1 \e3 1/2 iku} - 1/6 \cdot 1/6 \text{ \ar}
\]

\[
= 2 \text{ \ar2-gal 5 \ar2 49 \bur3} 5 \text{ 1/8 iku} 5 1/2 \text{ \ar} 1 2/3 \text{ \gin} - 1 2/3 \text{ \gin}.
\]

§4.5. A 5443, an Old Akkadian square-side-and-area exercise with decimal numbers (figure 11)

§4.5.1. The very small clay tablet ZA 74, p. 60, A 5443, is an Old Akkadian square-side-and-area

\[
\text{Fig. 11. ZA 74, p. 60, A 5443. An Old Akkadian square-side-and-area exercise with numbers that don't make sense.}
\]
exercise in which the numbers don't make sense. (The given length number is also strangely written with one of the signs for 1 gešu = 10 geš₂ misplaced.) It is potentially important to note that both the length number and the area number in A 5443 have, partially, the form of “funny numbers”. Thus, the given length number contains the digit 3 written 5 times, and the area number begins with the digit 1 repeated three times (1 šar₂-gal 11 šar₂; cf. DPA 36 in fig. 7 above, where the given length number is written with the digit 1 repeated five times.) Actually, since 1 nikkas = 1/4 ninda = 3 cubits, the given length number can be expressed as a number with the digit 3 repeated five times, namely as 33 geš₂ 33 ninda 3 cubits!

§4.5.2. The area of a square with the side 33 geš₂ ninda is 36 šar₂ 18 bur₃. Therefore, a square with the side 33 geš₂ 33 ninda 3 cubits, as in A 5443, cannot have the indicated area,

\[ A = 1 \text{ šar}_2\text{-gal} 11 \text{ šar}_2 27 \text{ bur}_3 5 \text{ iku} \frac{1}{2} \text{ šar} 2\frac{2}{3} \text{ gin}_2 5 \text{ gin}_2 <\text{tur}>. \]

A possible partial explanation of the strange numbers in A 5443 is that they may be an example of experimentation with the traditional Sumerian number systems, in an attempt to make them decimal.


§4.5.3. Therefore, suppose that in A 5443, contrary to the normal situation,

\[ \begin{align*}
1 \text{ geš}_u &= 10 \text{ geš}_2 = 1,000 \\
1 \text{ geš}_2 &= 100.
\end{align*} \]

Then 33 geš₂ 33 ninda 1 nikkas in this text means 3,333 \(1\frac{1}{4}\) ninda, and

\[ \text{sq. 3,333 n. 3 c.} = (\text{sq. 3,333} \cdot \frac{1}{2} \cdot \	ext{3,333} + \frac{1}{16}) \text{ sq. n.} = (11,108,889 + 1,666 \frac{1}{2} + \frac{1}{16}) \text{ sq. n.} = 11,110,555 \frac{1}{2} \text{ sq. n.} = 111,105 \text{ iku} 55 \frac{1}{2} \text{ šar} 3 \frac{2}{3} \text{ gin}_2 5 \text{ gin}_2 <\text{tur}>. \]

§4.5.4. Comparing this result with the area number recorded on A 5443, one is led to the tentative conclusion that in this text the large units of area measure may have had the following decimal values:

\[ \begin{align*}
1 \text{ šar}_2\text{-gal} &= 100,000 \text{ iku} \\
1 \text{ šar}_u &= 10,000 \text{ iku} \\
1 \text{ šar}_2 &= 1,000 \text{ iku}, \\
1 \text{ bur}_u &= 100 \text{ iku}, \\
1 \text{ bur}_3 &= 10 \text{ iku}.
\end{align*} \]

Under these assumptions, the area number recorded in A 5443 can be interpreted as meaning

\[ A = 111,275 \text{ iku} \frac{1}{2} \text{ šar}_2 3 \frac{2}{3} \text{ gin}_2 5 \text{ gin}_2 <\text{tur}>. \]

§4.5.5. Clearly, several of the initial and final digits in the computed area number are the same as the corresponding digits in the recorded area number. Unfortunately, the digits in the middle are not the same in the computed and the recorded area numbers. Presumably, the lack of agreement is due to some counting error, but an attempt to establish the precise nature of that error has not been successful. However, at least the last part of the recorded area number is correct, since

\[ \text{sq. 1 nikkas} = \frac{1}{16} \text{ šar} = 3 \frac{2}{3} \text{ gin}_2 5 \text{ gin}_2 <\text{tur}>. \]

It is interesting to note, by the way, that the given length number in A 5443 can be interpreted as

\[ 3,333 \text{ ninda 3 cubits} = 3,333 \frac{1}{3} \text{ ninda} - 1 \text{ cubit}. \]

This would be an explanation for the four initial digits 1 in sq. 3,333 n. 3 c., since

\[ \text{sq. 3,333 \frac{1}{3} \text{ ninda}} = \text{sq.} \left( \frac{1}{3} \cdot 10,000 \text{ n.} \right) = \frac{1}{9} \cdot 100,000,000 \text{ sq. n.} = 11,111,111 \frac{1}{9} \text{ sq. n.}. \]

§4.6. ZA 74, p. 65, A 5446, two Old Akkadian square-side-and-area assignments without answer (figure 12)

§4.6.1. ZA 74, p. 65, A 5446 is another Old Akkadian mathematical text published by Whiting 1984.
It appears to be a teacher’s note to himself that he has handed out two square-side-and-area exercises as assignments to two named students, Meluha and Ur-Ishtar. One of the assignments (# 2) is the one to which the answer is given in DPA 37 (§4.4 above). In the other assignment (# 1), the given length of the square-side is

\[28 \text{šar}_2 \cdot 1 \text{šu-du}_{3\text{-}a} = 30 \text{šar}_2 \cdot \frac{1}{15} \text{šar}_2 - 1 \text{šu-du}_{3\text{-}a}.\]

§4.6.2. The area of the square in A 5446 # 1 can have been computed in a few simple steps, just like the areas of the squares in DPA 36-37. The first step would probably have been to compute

\[\text{sq. } (30 \text{šar}_2 \cdot \frac{1}{15} \cdot 30 \text{šar}_2) = 30 \text{šar}_2\text{-kid} - 2 \cdot 2 \text{šar}_2\text{-gal} + 8 \text{šar}_2\text{-gal} = 26 \text{šar}_2\text{-kid} 8 \text{šar}_2\text{-gal}.\]

§4.6.3. For the second step of the computation, it would have been necessary to know the following relations:

- 1 \text{šu-du}_{3\text{-}a} = \frac{1}{6} \text{seed-cubit} = \frac{1}{6} \cdot \frac{1}{6} \text{ninda},
- 1 \text{n.} \cdot 1 \text{šu-du}_{3\text{-}a} = \frac{1}{6} \cdot 10 \text{šar} = 1 \frac{2}{3} \text{gin}_2,
- 1 \text{geš}_2 \text{n.} \cdot 1 \text{šu-du}_{3\text{-}a} = 1 \text{geš}_2 \cdot 1 \frac{2}{3} \text{gin}_2 = 1 \frac{2}{3} \text{šar},
- 1 \text{šar}_2 \cdot 1 \text{šu-du}_{3\text{-}a} = 1 \text{geš}_2 \cdot 1 \frac{2}{3} \text{šar} = 100 \text{išu} = 1 \text{iku},
- 1 \text{šu-du}_{3\text{-}a} \cdot 1 \text{šu-du}_{3\text{-}a} = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \text{ninda} = \frac{2}{3} \text{gin}_2\text{tur} \text{and} \frac{1}{6} \text{of } \frac{2}{3} \text{gin}_2\text{tur}.

§4.6.4. Clearly, then, the area of the square in A 5446 # 1 would have been found to be

\[26 \text{šar}_2\text{-kid} 8 \text{šar}_2\text{-gal} - 2 \cdot 28 \text{iku} + 2 \frac{2}{3} \text{gin}_2\text{tur} \text{and} \frac{1}{6} \text{of } \frac{2}{3} \text{gin}_2\text{tur}.

In Whiting 1984, the length of the given side of the square in A 5446 # 1 is expressed in sexagesimal numbers in place value notation as 27 59 59;58 20 ninda. Whiting wisely refrained from computing the square of this sexagesimal number and then trying to convert the result into Old Akkadian area numbers. It would have been more laborious than using the direct method indicated above.

§4.6.5. It should be noted that in the case of A 5446 # 1, the side of the square can be interpreted as a “fractionally and marginally contracted length number“. The side of the square in DPA 36 (§4.3) is a “fractionally and marginally expanded length number“, and the side of the square in DPA 37 is a “fractionally expanded and marginally contracted length number“. This can hardly be a coincidence. Instead, it is clear that the square-side-and-area problems in A 5446 and DPA 36-37 are based on three closely related but different variants of a clever geometric construction. The interesting conclusion of this observation must be that some anonymous mathematician in the Old Akkadian period (c. 2340-2200 BC) knew how to make use of one of the most fundamental mathematical tools, namely the systematic variation of a basic idea.

§4.6.6. It is not only the switching between expansions and contractions that catches the eye in the three mentioned texts. It is also the use of three different small units of length as the marginally added or subtracted numbers, namely the ŠU.BAD = \frac{1}{4} \text{seed-cubit} = \frac{1}{4} \cdot \frac{1}{6} \text{ninda in DPA 36}, the seed-cubit = \frac{1}{6} \text{ninda in DPA 37}, and the šu-du_{3\text{-}a} = \frac{1}{6} \text{ninda in A 5446}. In addition to this, in A 5443 (section 3.5), a marginally added unit of length is the nikkas = \frac{1}{4} \text{ninda. It was clearly the Old Akkadian teachers’ aim to teach their students how to handle all possible difficulties that could arise in the use of the Old Akkadian systems of length and area numbers.
§4.7. MAD 5, 112, an Old Akkadian text with three assignments (figure 13)

§4.7.1. The next text, MAD 5, 112 (Ashm. 1924.689), is inscribed on the obverse with two large length numbers, plus a name, Ur-Ištaran, perhaps the same Ur-Ištaran as the student who got an assignment on A 5446 and wrote his answer on DPA 37. On the reverse is recorded a very large area number. Powell 1976, p. 429, interpreted the text as giving the area of a very large rectangle, with the sides of the rectangle recorded on the obverse of the clay tablet, and wrote the following comment:

Ur-Ištaran was doubtless an unhappy little Sumerian when he recited the result of his computation which we find on the reverse of the tablet, for, the correct solution is 1, 1, 36, 16, 49, 41;26, 40 šar (or square nindan), but the number computed by the student apparently equals 3, 50, 0, 38, 46, 0;16, 40 šar.

§4.7.2. Whiting 1984, on the other hand, realized that MAD 5, 112, could be a parallel to A 5446, and therefore interpreted the text on the obverse as two assignments, the first to an individual whose name has been left out, the other to Ur-Ištaran. However, he conceded that

The area given on the reverse does not help resolve the interpretation of the obverse since it represents neither the product of the two numbers nor the square of either of them.

§4.7.3. Actually, as will be shown below, the area number on the reverse is (very close to) the square of a large length number (that it is not an exact square is probably due to a small computational error). Therefore, the correct interpretation of the text seems to be that there are two questions to square-side-and-area exercises on the obverse of MAD 5, 112, and one answer to an unrelated square-side-and-area exercise on the reverse.

§4.7.4. The given length number in assignment # 1 is bigger than all the length numbers in DPA 36-37 and A 5446:

\[ 1 \text{ šar₂-gal } 4 \text{ geš₂ ninda } 4 \text{ kuš₃ numun} \]
\[ = \text{ appr. } 1,296 \text{ km } + 1,440 \text{ m } + 4 \text{ m}. \]

The area of a square with this side would be immense. Note, in particular, that

\[ \text{sq. } 1 \text{ šar₂-gal ninda} \]
\[ = 2 \cdot 60^4 \text{ bur₃} \]
\[ = 2 \cdot 60^4 \cdot \text{ appr. } 64,800 \text{ square meters} \]
\[ = \text{ appr. } 1,680,000 \text{ sq. km}. \]

§4.7.5. The length number in assignment # 1 is 1 šar₂-gal 4 geš₂ ninda 4 kuš₃ numun. It is composed of one large length number, one of intermediate size, and one that is quite small. It is likely that the teacher’s intention was that the square expansion rule would be used twice for the computation of the area of a square with this side.

§4.7.6. The length number in assignment # 2 is 1 šar₂ 1 geš₂ 32 ninda 1 kuš₃ numun, which is again the sum of one large, one intermediate, and one small length number.

§4.7.7. The area number recorded in assignment # 3 on the reverse of MAD 5, 112, is

\[ A = 7 \text{ šar₂-kid } 40 \text{ šar₂-gal } 7 \text{ šar₂ } 17 \text{ bur₃ } 1 \text{ eš₃ } 3 1/2 \text{ iku aš₃ } 10 \text{ šar } 16 \text{ gin₂ } 2/3 <\text{gin₂} >. \]

This area number can be shown to be the answer to a square-side-and-area exercise. It is possible that the teacher’s instruction to one of his students was to find the side of a square of this area. If that is so, then the
method the student was assumed to use must have been a metro-mathematical variant of the “square side rule” often used in both Old and Late Babylonian mathematical texts to compute the sides of squares with given areas, or the square-roots of given non-square integers (cf. the detailed discussion in Friberg 1997, §8.)

§4.7.8. The idea behind the square side rule is simply an application, repeated as many times as necessary, of the square expansion rule in reverse. Therefore, the same kind of geometric model that can be used to explain the computation in several steps of the area of a square can also be used to explain the computation in several steps of the side of a square. Figure 14 demonstrates the first three steps of the computation of the side of the square with the area given in the assignment on the reverse of MAD 5, 112.

§4.7.9. Below are indicated the successive steps of the proposed computation of the square side.

\[
\begin{align*}
A & = 7 \text{ šār}_2 \text{-kid } 40 \text{ šār}_2 \text{-gal } 7 \text{ šār}_1 17 \text{ bur}_3 1 \text{ eš}_3 \quad 3 \frac{1}{2} \text{ iku ašša } 10 \text{ šār } 16 \text{ gin}_2 \frac{2}{3} \text{ <gin}_2 > \quad 1a. \\
A & = \text{ appr. } 7 \text{ šār}_2 \text{-kid } 30 \text{ šār}_2 \text{-gal ašša } 7 \text{ šār}_2 \text{-gal } 30 \text{ šār}_2 \text{ bur}_3 = 3 \text{ šār}_2 \text{-gal } 45 \text{ šār}_2 = 1 \text{ šār}_2 \text{ šār } = \text{ sq. } 15 \text{ šār}_2 \text{ n.} \quad 1b. \\
D_1 & = A - \text{ sq. } s_1 = (\text{ appr. }) 10 \text{ šār}_2 \text{-gal ašša} \quad 1c. \\
D_1 / 2s_1 & = (\text{ appr. }) 10 \text{ šār}_2 \text{-gal ašša } / 30 \text{ šār}_2 \text{ n. } = 10 \text{ šār}_2 \text{ bur}_3 / 30 \text{ šār}_2 \text{ n. } = 1 \text{ eš}_3 / 1 \text{ n. } = 10 \text{ geš}_2 \text{ n.} \quad 2a. \\
D_1 & = A - \text{ sq. } s_2 = (\text{ appr. }) 4 \text{ šār}_2 \text{ ašša} \quad 2b. \\
D_2 & = A - \text{ sq. } s_3 = (\text{ appr. }) 4 \text{ šār}_2 \text{-gal ašša} \quad 2c. \\
D_2 / 2s_2 & = (\text{ appr. }) 4 \text{ šār}_2 \text{-gal ašša } / 30 \text{ šār}_2 \text{ n. } = 4 \text{ geš}_2 \text{ bur}_3 / 30 \text{ šār}_2 \text{ n. } = 4 \text{ n.} \quad 3a. \\
D_3 & = A - \text{ sq. } s_4 = (\text{ appr. }) 15 \text{ šār}_2 \text{ geš}_2 \text{ n. } \quad 3b. \\
D_3 / 2s_3 & = (\text{ appr. }) 15 \text{ šār}_2 \text{ geš}_2 \text{ n. } / 30 \text{ šār}_2 \text{ n. } = 15 \text{ šār}_2 \text{ bur}_3 / 30 \text{ šār}_2 \text{ n. } = 15 \text{ šār}_2 \text{ eš}_3 \quad 3c. \\
E_3 & = \text{ sq. } s_5 - A = (\text{ appr. }) 3 \frac{1}{2} \text{ šār} \quad 4c. \\
E_4 / 2s_4 & = (\text{ appr. }) 5 \text{ bur}_3 / 30 \text{ šār}_2 \text{ n. } = 5 \text{ šār } / 1 \text{ geš}_2 \text{ n. } = 1 \frac{1}{12} \text{ n. } = 1 \frac{1}{9} \text{ kušša} \quad 4a. \\
E_5 / 2s_5 & = (\text{ appr. }) 1 \frac{1}{2} \text{ iku } / 30 \text{ šār}_2 \text{ n. } = 5 \text{ šār } / 1 \text{ šār}_2 \text{ n. } = 5 \text{ n. } / 1 \text{ šār}_2 = 1 \frac{1}{2} \text{ šu-si } \quad 5a. \\
E_6 / 2s_6 & = (\text{ appr. }) 7 \text{ šār}_2 \text{-kid } 40 \text{ šār}_2 \text{-gal } 7 \text{ šār}_2 17 \text{ bur}_3 1 \text{ eš}_3 3 \frac{1}{2} \text{ iku ašša } 13 \frac{1}{2} \text{ šār } 9 \frac{2}{3} \text{ gin}_2 \text{ tur} \quad 5b. \\
\text{ gin}_2 \text{ tur} & = 5 \text{ šār}_2 \text{-kid } 40 \text{ šār}_2 \text{-gal } 7 \text{ šār}_2 17 \text{ bur}_3 1 \text{ eš}_3 3 \frac{1}{2} \text{ iku ašša } 13 \frac{1}{2} \text{ šār } 9 \frac{2}{3} \text{ gin}_2 6 \quad 5c.
\end{align*}
\]

§4.7.10. Here the computation comes to an end. In the next step, one would have to subtract $\frac{5}{6}$ of $\frac{1}{6}$ of a barley-corn from the approximate square side $s_5$, and there would still not be an exact fit. Therefore, the recorded area number on the reverse of the text MAD 5, 112, is not a perfect square, although it is very close to one. Apparently, the scribe who wrote the text made a small error near the end of his computation of the square of the side $s = 15 \text{ šār}_2 10 \text{ geš}_2 \text{ ninda} - 1 \text{ kušša} 1 \frac{1}{2} \text{ šu-si}$. The precise cause of the error is difficult to establish.

In place value numbers $s = 15 \text{ šār}_2 10 \text{ geš}_2 \text{ ninda} - 1 \text{ kušša} 1 \frac{1}{2} \text{ šu-si} = 15 \frac{10}{4} \text{n.} - .05 \text{ n.} = 15 \frac{10}{4} \text{n.} = 15 \frac{10}{3} \text{gin}_2 \text{ tur}$. The area number recorded on the reverse of MAD 5, 112, corresponds, in sexagesimal place value numbers, to $3 \text{ 50 } 03 \text{ 38 } 46 \text{ 03 } 39 \text{ 45 } 50 \text{ 25 šār}$.

Note that a number of this form cannot be an exact square of a sexagesimal number. Indeed, if $a = b \cdot 60 + 10$ $16 \text{ 40}$, then $64 \cdot a = 64 \cdot b \cdot 60 + 3 \text{ 42}$ is a sexagesimal number with the last place 42, and no exact square of a sexagesimal number can have the last place 42.

§4.7.11. An intriguing extra twist is the astonishing circumstance that the side $s_2 = 15 \text{ šār}_2 10 \text{ geš}_2 \text{ninda}$ is not just an arbitrarily chosen length number. Indeed, the number 15 $\text{ šār}_2 10 \text{ geš}_2$ can be factorized as

$15 \text{ šār}_2 10 \text{ geš}_2 \text{ n. } = 10 \text{ geš}_2 \cdot 1 \text{ geš}_2 \text{ n. } = 10 \text{ geš}_2 \cdot 13 \cdot 7 \text{ n.}$

$= (1 + 1\frac{1}{12}) \cdot (1 + 1\frac{1}{6}) \cdot 12 \text{ šār}_2 \text{ ninda}.$

§4.7.12. This is certainly not a coincidence, since the factorization shows that 15 $\text{ šār}_2 10 \text{ geš}_2$ is a good example of an “almost round number”. By definition, an almost round number is a number which is close to a round number and simultaneously equal to another round number multiplied by one or two factors of the kind $(1 + \frac{1}{n})$, where $n$ is a small regular sexagesimal integer (note that 15 $\text{ šār}_2 10 \text{ geš}_2$ is very close to the round number 15 $\text{ šār}_2$, since 10 $\text{ geš}_2$ is only $\frac{1}{10}$ of 15 $\text{ šār}_2$). It is easy to find examples of almost round area numbers in proto-cuneiform field-area and fieldsides texts from Uruk around the end of the 4th millennium BC (see Friberg 1997/1998, fig. 2.1). It is also easy to find examples of almost round area numbers in Old Babylonian mathematical texts. See, for instance, Friberg nd, ch. 1.2 (MS 2831, a series of five squaring exercises), and ch. 8.1.b (MS 2107 and MCT 44 [YBC 7290], two computations of the area of a trapezoid). Two further examples of almost round numbers in Old Akkadian computations of areas are DBA 3 and OIP 14.
A skeptical reader may object that every given number is very close to an exact square, namely the square of its approximate square root. However, that objection is not valid in the case of the area number recorded on the reverse of MAD 5, 112, since, as was shown above, that number is very close to the square of 15\(\frac{1}{2}\) ar\(\text{a}2\) ge\(\text{a}2\) 2 ninda - 1 kuš\(\text{a}3\) which is a very special kind of length number. As remarked already, 15\(\frac{1}{2}\) ar\(\text{a}2\) ge\(\text{a}2\) 2 ninda is an almost round length number. In addition, just like the two length numbers recorded on the obverse of the clay tablet, the length number 15\(\frac{1}{2}\) ar\(\text{a}2\) ge\(\text{a}2\) 4 ninda is composed of one large length number, one of intermediate size, and one that is quite small. The whole number 15\(\frac{1}{2}\) ar\(\text{a}2\) ge\(\text{a}2\) 4 ninda - 1 kuš\(\text{a}3\)\(\frac{1}{2}\) śu\(\text{a}3\) can be understood as a two times expanded and two times contracted length number. Note also that the second term is 90 times smaller than the first term, the third term is 150 times smaller than the second term, the fourth term is 48 times smaller than the third term, and the fifth term is 60 times smaller than the fourth term.
§4.9. *OIP* 14, 70, an *ED IIIb* table (Adab) of areas of small squares (figure 16)

§4.9.1. Whiting 1984, p. 64, even thought that he could find evidence for counting with sexagesimal place value numbers in the table of small squares *OIP* 14, 70 (cf. Edzard 1969, 101-104), an *ED IIIb* text from Adab. Consequently, he ended his paper with the following statement:

*In summary, the evidence provided by Powell supplemented by that presented here prompts me to state with confidence that sexagesimal place value notation was being used to perform calculations in the Old Akkadian period and that instruction in these techniques was being carried out at Lagash/Girr and probably at Nippur. Less strong, but still significant, is the evidence that the use of sexagesimal place notation was known before the Sargonic period, especially for the expression of fractions.*

§4.9.2. The brief argument which Whiting uses in order to prove that sexagesimal place value notation was used by the author of *OIP* 14, 70, goes as follows:

These two lines (lines 10 and 14) give the squares of 5 cubits and 7 cubits respectively. (in sexagesimal notation) 

\[(0.25 \text{nindan})^2 = 10 \text{ gin} \frac{25}{102} \text{ sar} \quad \text{and} \quad (0.35 \text{nindan})^2 = 20 \text{ gin} \frac{25}{102} \text{ sar}.

It can be seen that the quantities written in these lines, 10 gin\(\frac{25}{102}\) and 13.3 gin\(\frac{25}{102}\), exactly express the correct answers in sexagesimal place notation.

§4.9.3. The argument is, of course, not compelling. It only expresses the trivial truth that the correct values of the squares, expressed in sexagesimal place value numbers, are equal to the correct values of the same squares expressed in Old Akkadian fractional notations.

§4.9.4. *OIP* 14, 70, is extensively discussed in Friberg nd, app. 1. It is shown there how all the values in the table of squares can have been computed with departure from the result recorded in the first line of the table, namely (in terms of exchange-minas and shekels of exchange-minas)

\[
\text{sq. 1 cubit} = 1 \frac{1}{4} \text{ sa}_{10} \text{ ma-na} = 1 \text{ sa}_{10} \text{ ma-na} 15 <\text{sa}_{10}> \text{ gin}_{2}.
\]

§4.9.5. That result, in its turn, can have been obtained as follows:

\[
\begin{align*}
1 \text{ reed} &= 1/2 \text{ ninda} \\
1 \text{ cubit} &= 1/6 \text{ reed}, \\
1 \text{ sq. reed} &= 1/4 \text{ sar} \\
&= 15 \text{ gin}_{2}, \\
1 \text{ reed} \cdot 1 \text{ cubit} &= 15 \text{ gin}_{2}/6 \\
&= 2 1/2 \text{ gin}_{2}, \\
1 \text{ sq. cubit} &= 2 1/2 \text{ gin}_{2}/6 \\
&= 2 \frac{1}{2} \text{ gin}_{2}/6, \\
&= 7 \frac{1}{2} \text{ sa}_{10} \text{ ma-na}/6 \\
&= 1 \frac{1}{4} \text{ sa}_{10} \text{ ma-na} \\
&= 1 \text{ sa}_{10} \text{ ma-na} 15 <\text{sa}_{10}> \text{ gin}_{2}.
\end{align*}
\]

§4.10. *TSŠ 50* and 671, metric division exercises from Šuruppak (ED IIIa; figure 17)

§4.10.1. There are two known Old Sumerian division exercises from Šuruppak in the Early Dynastic IIIa period, about the middle of the 3rd millennium BC (see Friberg 1990, §4.2). One of them is *TSŠ 50*, where the question is how many men can receive rations of 7 sila\(3\) each from the barley in a granary. The answer is given, correctly, in non-positional sexagesimal numbers. The details of the solution procedure are not provided.

§4.10.2. A related text, also from Šuruppak, is *TSŠ 671*. Both *TSŠ 50* and 671 were first published in Jestin 1937. Photos of the clay tablets can be found in Høyrup 1982.
§4.10.3. TS§50 and 671 are both concerned with the same division exercise. The unstated question can be reconstructed as follows: All the barley contained in 1 ‘granary’ is to be divided into rations, so that each man (lu2) or worker (guru) receives 7 sila3. How many men get their rations?

§4.10.4. The exact answer given in TS§50 is 45 šar2 42 geš2 51, with 3 sila3 left over (šu tag4). Counting backwards, one finds that the barley contained in 1 ‘granary’ must have been

\[ 7 \text{ sila}_3 \cdot 45 \text{ šar}_2 42 \text{ geš}_2 51 + 3 \text{ sila}_3 \]

\[ = 5 \text{ šar}_2 \text{-gal} 20 \text{ šar}_2 \text{ sila}_3. \]

§4.10.5. This result agrees fairly well with the known fact that in later cuneiform texts, both Sumerian and Babylonian, gur7 ‘granary’ was the name of a very large capacity unit, equal to 1 šar2 gur. The gur was another large capacity unit, equal to 5 geš2 (300) sila3 in Old Babylonian mathematical texts, but of varying size in Sumerian administrative texts, depending on from what site and from which period the texts originate. The size of the sila3 was also varying but was probably always roughly equal to 1 liter.

§4.10.6. What TS§50 seems to tell us is that in Šuruppak, in the middle of the 3rd millennium BC, 1 gur7 was equal to 1 šar2 gur, with each gur equal to 5 geš2 20 (320) sila3. Actually, however, only two kinds of gur are documented in texts from Šuruppak, in some texts the gur ma₇, ‘mighty gur’, equal to 8 geš2 sila3, in other texts the lid₂-ga, equal to 4 geš₂ sila3. The reason for the curious appearance in TS§50 of what appears to be a gur of 5 geš₂ 20 sila3 is unknown, but it can possibly be explained as follows. (cf. Friberg 1997, §§7b, 7e, and fig. 7.1).

§4.10.7. In Old Babylonian mathematical problem texts and mathematical tables of constants, sila3 measures of various sizes appear, explicitly or implicitly. The size of a sila₃ of a given kind is indicated by its “storing number” (naₗ₆₃ₙ₆₃), which can be explained as the number of sila₃ of that particular kind contained in a volume unit. One common type of sila₃ was the “cubic sila₃”, a cube of side 6 fingers (;01 ninda), with the storage number “5” (meaning 5 00 sila₃ per volume-shekel). Another type of sila₃, known only from a table of constants, was the sila₃ of the ‘granary’ (k₆₆ₗ₆₃ₙ₆₃) with the storing number 7 30 (7 30 sila₃ per volume-shekel). Clearly the size of the granary sila₃ must have been only 2/3 of the size of an ordinary cubic sila₃, since 5 00 = 2/3 of 7 30. Therefore, 1 gur₇ of granary sila₃ would be as much as only 5 geš₂ 20 cubic sila₃, the number of sila₃ counted with in TS§50 and 671. This strange counting with two different storing numbers can be explained if it is assumed that, for some reason, contents of granaries were habitually counted in granary sila₃ while rations were counted in cubic sila₃ (unfortunately, nothing is known about why there existed sila₃-measures of several different sizes).

§4.10.8. Powell thought that the answers in TS§50 and 671 had been computed by use of sexagesimal numbers in place value notation. Thus, in HistMath 3 (1976), 433, he suggests that the correct answer in TS§50 might have been obtained by the following process, using an accurate reciprocal of 7:

\[
\begin{align*}
(1) & \quad 5 20 00 00 \cdot 08 34 17 08 = 45 42 51;22 40, \\
(2) & \quad 45 42 51 \cdot 7 = 5 19 59 57, \\
(3) & \quad 5 20 00 00 - 5 19 59 57 = 3.
\end{align*}
\]

Here the sexagesimal fraction ;08 34 17 08 is a fairly accurate approximation to the reciprocal of 7, since

\[
7 \cdot 08 34 17 08 = ;59 59 59 56 = 1 - ;00 00 00 04.
\]
§4.10.9. Powell also thought that the incorrect answer in $TS^5$671 (45 šar$_2$ 36 ge$_2$) could be explained as the result of using the slightly incorrect value 08 33 for the reciprocal of 7. Indeed,

\[
\begin{align*}
520000 & :08 33 = 520 \cdot 8 33 \\
& = 42 45 00 + 2 51 00 \\
& = 45 36 00.
\end{align*}
\]

§4.10.10. On the other hand, Melville 2002, 237-252, was able to give a much more convincing explanation of how the answers in $TS^5$ 50 and 671 can have been computed by an ED IIIa school boy without recourse to sexagesimal numbers in place value notation. Although Melville failed to realize it, the solution algorithm he proposes is closely related to a solution algorithm in terms of non-positional decimal numbers used in $MEE$ 3, 74 (TM.75.G 1392), a metric division exercise from Ebla, nearly as old as the texts from Suruppak (Friberg 1986). Somewhat simplified and refined, Melville's explanation goes as follows in the case of $TS^5$50:

<table>
<thead>
<tr>
<th>barley</th>
<th>men receiving</th>
<th>remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>rations</td>
<td></td>
</tr>
<tr>
<td>1) 1 ban$_2$ = 10 sila$_3$</td>
<td>1</td>
<td>3 sila$_3$</td>
</tr>
<tr>
<td>2) 1 barig = 6 ban$_2$</td>
<td>8</td>
<td>4 sila$_3$</td>
</tr>
<tr>
<td>3) 1 gur mah = 8 barig</td>
<td>1 ge$_2$ 8</td>
<td>4 sila$_3$</td>
</tr>
<tr>
<td>4) 10 gur mah</td>
<td>11 ge$_2$ 25</td>
<td>5 sila$_3$</td>
</tr>
<tr>
<td>5) 1 ge$_2$ gur mah</td>
<td>1 šar$_2$ 8 ge$_2$ 34</td>
<td>2 sila$_3$</td>
</tr>
<tr>
<td>6) 10 ge$_2$ gur mah</td>
<td>11 šar$_2$ 25 ge$_2$ 42</td>
<td>6 sila$_3$</td>
</tr>
<tr>
<td>7) 40 ge$_2$ gur mah</td>
<td>45 šar$_2$ 42 ge$_2$ 51</td>
<td>3 sila$_3$</td>
</tr>
</tbody>
</table>

the corresponding remainder calculations:

1) $10 \text{ sila}_3 = 1 \text{ ration} + 3 \text{ sila}_3$
2) $6 \cdot 3 \text{ sila}_3 = 2 \text{ rations} + 4 \text{ sila}_3$
3) $8 \cdot 4 \text{ sila}_3 = 4 \text{ rations} + 4 \text{ sila}_3$
4) $10 \cdot 4 \text{ sila}_3 = 4 \text{ ban}_2 = 4 \text{ rations}$

§4.10.12. The reason for the mistake may be that the normal size of a ration was 1 ban$_2$ (10 sila$_3$), and that the author of $TS^5$ 671 for a moment forgot that in this exercise it was supposed to be instead only 7 sila$_3$.

§5. Conclusion
§5.1. The detailed discussion above makes it clear that all computations in known mathematical cuneiform texts from the third millennium can be explained without the use of sexagesimal numbers in place value notation. Besides, explanations in terms of sexagesimal place value numbers severely obscure the fine points of the texts. It is like cracking nuts with a sledge hammer.

§5.2. The discussion in §§2-3 of Old Akkadian metric division texts shows that Old Akkadian mathematicians were familiar with the notion of regular sexagesimal numbers and suggests that they were familiar with the powerful idea of factorization algorithms. Similarly, the discussion in §§4.2-6 of Old Akkadian square-side-and-area texts suggests that Old Akkadian mathematicians were familiar with both funny numbers and almost round numbers, with the square expansion and square contraction rules, and with the powerful tool of a systematic variation of a basic idea.

§5.3. The discussion of $MAD$ 5, 112, rev. in section §§4.7-11 reveals that Old Akkadian mathematicians might have been familiar with an efficient algorithm for the computation of the side of a square of given area, and the discussion of IM 58045 in section §4.8 shows that they were familiar also with the interesting topic of epiphetration trapezoids.

§5.4. Finally, the discussion of ED IIIa metric division problems in §4.10 suggests that efficient metric division algorithms may have been known already in the ED III period.

§5.5. What is particularly important in this connection is that the mentioned crucial concepts and methods (with the possible exception of the metric division algorithms) also played important roles in Old Babylonian mathematics. Therefore, there can be little doubt that Old Babylonian mathematics had inherited many of its central ideas from its predecessors in the 3rd millennium BC.
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