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On the Alleged Counting with Sexagesimal Place Value Numbers in Mathematical Cuneiform Texts from the Third Millennium BC

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§1.1. Four different kinds of problems appear in the small corpus of known mathematical texts from the Old Akkadian (Sargonic) period, ca. 2340-2200 BC. In texts of the first kind, the area of a rectangle or a nearly rectangular quadrilateral is computed. Examples are *ArOr* 50, 1 (NBC 7017), *MAD* 4, 163 (AO 11404), 164 (AO 11405) and 166 (AO 11409) (all in Foster and Robson 2004), as well as *DPA* 34 and *OIP* 14, 116 (Ad 786; see Friberg nd, chapter A6.a). In texts of the second kind, here called "metric division exercises," the short side of a rectangle is computed when the area and the long side of the rectangle are given. To this category belong *DPA* 38-39, *TMH* 5, 65, and two privately owned texts, all described in §§2-3 below. In texts of the third kind, "square-side-and-area exercises," the area of a square is computed when the side is known. To this category belong *DPA* 36-37, *ZA* 74, p. 60 (A 5443) and 65 (A 5446) (both in Whiting 1984), and *MAD* 5, 112 obv. (Ash. 1924.689; Gelb 1970), all described in §§4.2-4.7 below. The only known text of the fourth kind is IM 58045 = 2N-T 600 (Friberg 1990, p. 541), a round hand tablet with a drawing of a partitioned trapezoid with a transversal of unknown length (§4.8 below).

§1.2. It has been claimed repeatedly by several authors (Powell, Whiting, and most recently Foster and Robson) that sexagesimal numbers in place value notation must have been used in the complicated computations needed to solve the problems stated in the Old Akkadian metric division exercises and square-side-and-area exercises, always without explicit solution procedures. The aim of the present paper is to show that it is easy to explain those computations in less anachronistic ways. Actually, all known mathematical exercises from the 3rd millennium BC are "metro-mathematical," in the sense that they are not simply concerned with relations between abstract numbers but rather with relations between measures for lengths, areas, volumes, capacities, etc. In support of this thesis, there is also a brief discussion below of *OIP* 14, 70, a table of areas of small squares from Adab, ED IIIb (§4.9), and of *TSŠ* 50 and 671, two metric division exercises from Šuruppak of the ED IIIa period (§4.10).

§1.3. The conclusion of the discussion in the present paper is that the earliest known firmly dated example of the use of sexagesimal numbers in place value notation is *YOS* 4, 293 (YBC 1793; see Powell 1976, and §4.1 below), a unique "scratch pad" from the Ur III period, ca. 2100-2000 BC, with totals of sexagesimal place value numbers representing traditional weight numbers.

§2. *Two privately owned Old Akkadian mathematical texts*

§2.1. *CMAA 016-C0005, an Old Akkadian metric division exercise of standard type (figure 1)*

§2.1.1. The clay tablet from the collection of the California Museum of Ancient Art, Los Angeles (director: J. Berman), shown in figure 1 below, is a small hand tablet inscribed on the obverse with a brief mathematical exercise. The similarity of this text to several previously known Old Akkadian mathematical texts makes it immediately clear that his text, too, is such a text.

§2.1.2. The statement of the problem (the question) is given in a condensed form in lines 1-2:

$9_{\text{geš2}}$ uš	*9 geš$_2$ the length,*
sag 1_{iku} aša$_5$	*the front: 1 iku field.*

What this means is that a rectangle has a given long side (called uš 'length') equal to 9 geš$_2$ = 9 · 60 (ninda) and a given area (referred to by the determinative [GAN$_2$=] aša$_5$, 'field') equal to 1 iku. The obliquely formulated problem is to find the short side (called sag 'front'). The answer is given in line 3:

2 kuš$_3$ 6 $^2/_3$ šu-si	*2 cubits 6 $^2/_3$ fingers.*

Fig. 1. CMAA 016-C0005. An Old Akkadian metric division exercise of standard type.

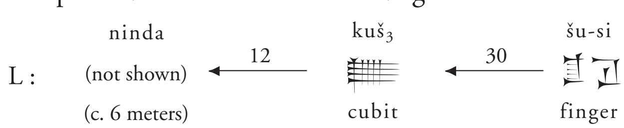
§2.1.3. The length units employed in this text are

the ninda (a measuring rod of about 6 meters),
the cubit (about $^1/_2$ meter),
the finger (about 1.7 cm).

The ninda was the main length unit, and the cubit and the finger were fractions of the ninda, with

1 ninda = 12 cubits,
1 cubit = 30 fingers.

Such relations between a series of units can conveniently be expressed in terms of a “factor diagram”:



§2.1.4. Thus, 2 cubits 6 $^2/_3$ fingers = 66 $^2/_3$ fingers = (66 $^2/_3$ ÷ (12 · 30)) ninda = $^5/_{27}$ ninda, and it follows that the long side of the rectangle is 2,916 times longer than the short side. Such an unrealistic relation between the sides of a field shows that this is an artificially constructed mathematical exercise.

§2.1.5. Being the main length unit, the ninda was often silently understood, as in line 1 of this text. (It was also implicitly understood in many Old Babylonian mathematical texts, and even in certain types of proto-cuneiform texts from the end of the 4th millennium BC). Closely associated with the ninda was the main area unit the iku = 100 square ninda.

§2.1.6. It is, of course, not known how the answer to the question in lines 1-2 of the present text was found. However, a simple and efficient solution algorithm that may have been used starts with a square of area 1 iku. In a number of steps, the square is replaced by a series of progressively longer rectangles of the same area, until in the last step a rectangle is found with the required length and with a front that is the answer to the problem. Here are the successive steps of a factorization algorithm of this kind in the case of CMAA 016-C0005:

1 iku	= 10 n. · 10 n.	a square with the side 10 ninda has the area 1 iku
	= 1 geš$_2$ n. · 1 $^1/_2$ n. 2 c.	a 6 times longer and 6 times more narrow rectangle of the same area
	= 3 geš$_2$ n. · $^1/_2$ n. $^2/_3$ c.	a 3 times longer and 3 times more narrow rectangle of the same area
	= 9 geš$_2$ n. · 2 c. 6 $^2/_3$ f.	a 3 times longer and 3 times more narrow rectangle of the same area

Hence, the answer is that the front is 2 cubits 6 $^2/_3$ fingers.

§2.1.7. Indeed, it is easy to check that

$^1/_6$ · 10 n. = 1 n. 8 c. = 1 $^1/_2$ n. 2 c.,
$^1/_3$ · 1 $^1/_2$ n. 2 c. = $^1/_2$ n. $^2/_3$ c.,
$^1/_3$ · $^1/_2$ n. $^2/_3$ c. = 2 c. 6 $^2/_3$ f.

The suggested solution algorithm is of the same kind as some known Old and Late Babylonian factorization algorithms used for similar purposes. For an interesting Late Babylonian example, see Friberg 1999a/2000a.

§2.1.8. It is appropriate to call an exercise of this kind a “metric division exercise”, since the object of the exercise is not to divide a number by another number, but to divide a given area by a given length.

§2.2. *ZA 94, 3, an Old Akkadian metric division exercise of a non-standard type (figure 2)*

§2.2.1. The Old Akkadian clay tablet shown in figure 2 below is a privately owned medium size hand tablet inscribed on the obverse with a brief mathematical exercise. It was published in Foster and Robson 2004. The hand copy in figure 2 is based on the copy of the text in *ZA* 94.

§2.2.2. The text is well preserved, with only a couple of erasures in lines 1-2 and small damaged regions in lines 7-8. Nevertheless, the interpretation of the text is difficult because there are some unusual number notations in lines 1-2, a lost number sign in line 7, and possibly an incorrect or miscopied number sign in line 6. The meaning of the signs in lines 8 and 9 is difficult to establish. In addition, the meaning of the phrase in lines 3-5 is far from clear.

Unfortunately, the present whereabouts of the text are unknown, so the hand copy cannot be collated. It is regrettable that high resolution photographic images of the tablet were not produced at the same time as the hand copy. It is not to be expected that an accurate hand copy can be produced of a difficult text that the copyist does not understand.

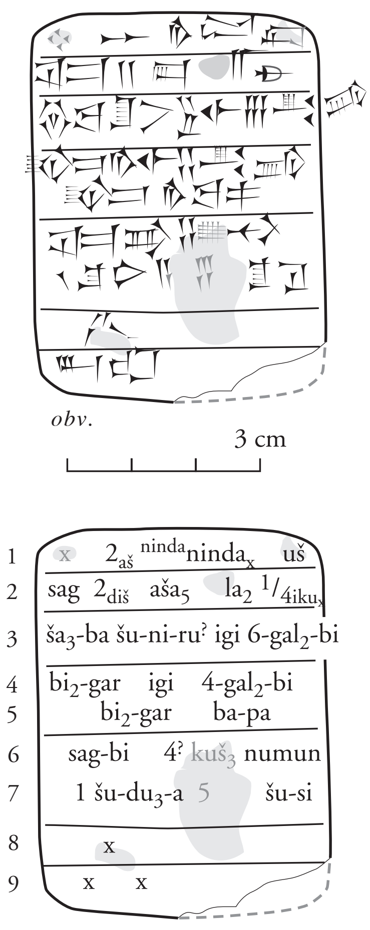


Fig. 2. ZA 94, 3. An Old Akkadian metric division exercise of a non-standard type.

§2.2.3. For all these reasons, only conjectural interpretations of the text can be suggested, and it is no wonder that the interpretation proposed below differs in several ways from the interpretation proposed, on rather loose grounds, by Foster and Robson.

§2.2.4. The statement of the problem (the question) is given in a condensed form in lines 1-2:

1.	(erasure) $2_{\text{aš}}$ $^{\text{ninda}}$ninda$_{\text{x}}$(DU) uš	*(erasure) 2(· 60)? ninda the length*
2.	sag $2_{\text{diš}}$ aša$_5$ (erasure) la$_2$ $^1/_{4\text{ikux}}$	*the front: $^2/_3$(iku)? less $^1/_4$(iku)?*

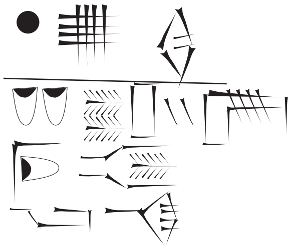
§2.2.5. Thus, the question begins in line 1 by stating that the length (of a rectangle) is $2_{\text{aš}}$ ninda. Here, $2_{\text{aš}}$ is a number sign in the form of two horizontal cuneiform wedges (aš), placed one after the other, presumably meaning either 2 or 2 · 60. A quick survey of other Old Akkadian or ED III texts mentioning length numbers has not turned up any parallel number notations for length measures. Normally in such texts, ones are denoted by vertical or slightly inclined cuneiform wedges, and sixties are denoted by slightly larger vertical cuneiform wedges, as for instance in *DPA* 38-39 (figs. 3, left, and 4), *BIN* 8, 24 and 147, and in *OIP* 14, 116 and 163. However, *TMH* 5, 65 (fig. 3, right) is an exception, since in line 1 of that text both ones and sixties are denoted by curviform (as opposed to cuneiform) horizontal aš signs. Therefore, it is possible that the horizontal number signs in line 1 of *ZA* 94, 3, are the cuneiform counterparts of the curviform number signs for ones or sixties in *TMH* 5, 65.

§2.2.6. In line 2, left, the length of the front (of the rectangle) is requested with the single word sag 'front', in the same oblique way as in line 2 of CMAA 016-C0005 (fig. 1). Then follows the information that the area of the rectangle is $^2/_{3\text{iku}}{}^?$ - $^1/_{4\text{iku}}{}^?$, with $^2/_{3\text{iku}}{}^?$ written as $2_{\text{diš}}$ aša$_5$, where $2_{\text{diš}}$ is a number sign in the form of two cuneiform vertical wedges (diš). At least, this is one possible reading of the area number.

Another possibility is to read $2_{\text{diš}}$ aša$_5$ as 2 iku, as it is done in Foster and Robson 2004. However, 2 iku is normally written with two horizontal aš-signs.

§2.2.7. The use of two curviform vertical diš-signs to denote $^2/_3$(?) šar is documented in ED IIIa house purchase contracts such as *PBS* 9 (1915) no. 3 i 3 ($1_{\text{aš@c}}$ $2_{\text{diš@c}}$ šar e$_2$-bi), and *TMH* 5, 75 obv. i 3 ($1_{\text{aš@c}}$ $2_{\text{diš@c}}$ šar e$_2$-bi), both from Nippur. It appears also in the table of small squares *OIP* 14, 70 (ED IIIb, Adab; fig. 16

below), where the square of 10 cubits is expressed as follows:



10 $kuš_3$ sa_2

$2_{diš@c}$ šar $2_{aš@t}$ gin_2

la_2 $1_{aš@c}$ sa_{10}

ma-na

What this means is that

10 cubits squared = $2/3$ šar 2 shekels - 1 exchange-mina,

where

1 exchange-mina = $1/3$ shekel.

§2.2.8. This equation can be explained as follows (cf. Friberg nd, app. 1, under fig. A1.4):

1 reed = $1/2$ ninda
1 sq. reed = $1/4$ sq. ninda = 15 (area-)shekels
1 sq. cubit = $1/36$ sq. reed = $1/36 \cdot 15$ shekels
= $1/12 \cdot 5$ shekels = $1/12 \cdot 15$ exchange-minas
= 1 $1/4$ exchange-mina
sq. (10 cubits) = $100 \cdot 1\ 1/4$ exchange-mina.
= 125 exchange-minas = $2/3$ šar 2 shekels - 1 exchange-mina

An explanation of the name sa_{10} ma-na 'exchange-mina' for $1/3$ shekel was proposed in Friberg 1999b, 133, namely that in the ED IIIa period silver was 180 times more expensive than copper, and that therefore, originally, copper was measured in terms of minas, while silver was measured in terms of the 180 times lighter exchange-minas. For a proof of the proposed value of silver relative to copper, see D. O. Edzard 1968. In Edzard's Table 1 (p. 21), there is a list of purchase prices for a number of fields. There is no fixed relation between purchase price and area. However, in Edzard's Table 3 (p. 22) there is a list of fees for the scribe in 8 ED IIIa field purchase contracts. There was clearly a fixed rate of fee to area, either 1 shekel of silver per $eše_3$ = 6 iku (texts 1, 3, 4, 7, 8) or 3 minas of copper per $eše_3$ (texts 2, 6, 9). Therefore, 1 shekel of silver was equal to 3 minas of copper, or, equivalently, 1 exchange-mina of silver = 1 mina of copper. The unit $sa_{10}+1_{aš}$ ma.na is attested in Edzard's text 7 = *RTC* 14 (p. 339): 1 ku_3 gin_2 $sa_{10}+1_{aš}$ ma-na = 1 $1/3$ shekel of silver as the scribe's fee for a field of the size 1 $eše_3$ 2 iku = 1 $1/3$ $eše_3$, and in text 8 = *RTC* 15 (p. 35): ku_3 $sa_{10}+1_{aš}$ ma-na = 1 exchange-mina of silver as the scribe's fee for a field of the size 2 iku = $1/3$ $eše_3$.

§2.2.9. Notations for fractions resembling some of those used in the ED IIIb Agade text *OIP* 14, 70, can be found also in *BIN* 8, 175 (=NBC 6915; Edzard 1968, no. 54), an Old Akkadian slave sale contract from Nippur. There the sum of 4 shekels of silver (4 ku_3 gin_2), $1/2$ shekel ($1/2$ ku_3 gin_2), $1/4$ shekel (ku_3 igi 4-gal_2-kam), 4 shekels, 2 shekels, and 2 exchange-minas (ku_3 $sa_{10}+2_{aš}$ ma-na-kam) is given as 11 $1/2$ shekels - 15 exchange-shekels, written as 11 $1/2$ la_2 $sa_{10}+15$ ku_3 gin_2. The summation shows that

10 $1/2$ $1/4$ shekels + 2 exchange-minas
= 11 $1/2$ shekels - 15 exchange-shekels.

This equation can be reduced to

2 $1/4$ exchange-minas = $1/2$ $1/4$ shekel,

or simply

1 exchange-mina = $1/3$ shekel.

§2.2.10. It is interesting to note that a mix of curviform and cuneiform number signs are used in *OIP* 14, 70 (fig. 16 below). Thus, curviform horizontal aš-signs are used in front of šar (square ninda), $kuš_3$ (cubit), gi (reed = $1/2$ ninda), and sa_{10} ma-na (exchange-mina), while cuneiform, slanting diš signs are used in front of gin_2 (shekel). For the basic fractions of the šar, both curviform and cuneiform number signs are used in *OIP* 14, 70, with curviform signs for $1/2$ and $2/3$, but a cuneiform sign for $1/3$. The curviform and cuneiform signs, respectively, for the three basic fractions are:

šanabi ("$2/3$"=$2_{diš(@c)}$) maš ("$1/2$"=$1_{diš(@c)}$+aš) šušana ("$1/3$"=$šu_2+1_{diš(@c)}$)

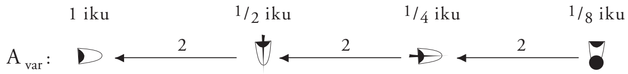
These basic fractions could also be used for fractions of the 60-shekel mina. See, for instance, *OIP* 14, 76, where the curviform signs are used for both $2/3$ and $1/3$, as fractions of a mina.

§2.2.11. So far, the cuneiform form above of the sign for $2/3$ seems to be documented only, possibly, in *ZA* 94, 3. For the sake of completeness, it must be remarked here that there were also other ways of writing $2/3$ in ED IIIb texts. The following two examples are taken from *OIP* 14, 48 and 49:

 (followed by the word ša-na-pi) 

Note that in the Old Akkadian mathematical texts in figs. 9, 11, and 13 below, the sign for $2/3$ is different again and is followed by -ša

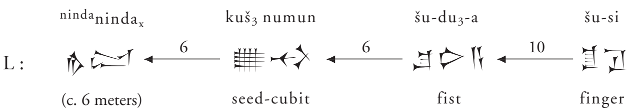
§2.2.12. The meaning of the last number sign in line 2 of *ZA* 94, 3 is possibly $1/4$ iku. See Powell, *ZA* 62, 217-220. According to Powell, the following notations for iku fractions, otherwise unknown, occur in *BIN* 8, nos. 49, 51, 112, 114, 120 (ED IIIb), and 189, 190, 195, 199 (Old Akkadian):



§2.2.13. The answer to the question in *ZA* 94, 3 is given in lines 6-7 of the text:

sag-bi $4^?$ kuš$_3$ numun	*Its front (is) $4^?$ seed-cubits*
1 šu-du$_3$-a $[5]^?$ šu-si	*1 fist, $[5^?]$ fingers.*

The relative sizes of the length measures mentioned here are given by the following factor-diagram:



§2.2.14. The text in lines 3-5 of *ZA* 94, 3 is, conceivably, part of either the solution algorithm or the answer. Its meaning is far from clear. Here is a partial translation of the obscure passage:

3.	ša$_3$-ba šu-du$_3$-ru?	*Inside it X.*
4.	igi 6-gal$_2$-bi bi$_2$-gar	*Its 6th-part set,*
	igi 4-gal$_2$-bi bi$_2$-gar	*its 4th-part set.*
5.	ba-pa	*It is found.*

§2.2.15. If everything with the text had been in order, it should now be possible to show that in *ZA* 94, 3 the given area is equal to the product of the given length and the computed front. However, this is easier said than done. Apparently, the one who wrote the text made a couple of mistakes. It is also possible that the hand copy of the text in *ZA* 94, 3, contains one or several mistakes, for instance a mistake in line 6, where what looks like a 4 in front of a small damaged kuš$_3$ sign may, conceivably, be a 1 in front of a larger kuš$_3$ sign.

§2.2.16. In any case, here follows a proposed reconstruction of the successive steps of the solution algorithm in *ZA* 94, 3, based on the assumption that the given area A is 2/3 iku - 1/4 iku, as suggested above, and on the further assumption that the given length u is 2(· 100)$^!$, rather than 2(· 60) <ninda>.

1. A/200 = (2/3 iku - 1/4 iku)/200 = (2/3 šar - 1/4 šar)/2
 = (40 - 15) area-shekels/2 = 12 1/2 area-shekels.
2. 12 1/2 area-shekels = 1/6 area-šar + 1/4 of 1/6 area-šar.
3. (1/6 area-šar + 1/4 of 1/6 area-šar) / 1 ninda
 = 1/6 ninda + 1/4 of 1/6 ninda = 1 1/4 seed-cubit.
4. Hence, the front s = A/u = 1 1/4 seed-cubit
 = 1 seed-cubit 1 šu-du$_3$-a 5 fingers.

§2.2.17. It is likely that the detour over the composite fraction 1/6 + 1/4 · 1/6 was necessary for the reason that it would not have been legitimate to speak about '12 1/2 shekels of a ninda'. Note that the use of a composite fraction of a similar type is documented in the Old Babylonian combined work norm exercise *MCT* 81 (YBC 7164) §7 (Friberg 2000b, 127). There, a man works '1/5 of a day throwing' (mud) and '2/3 of a day and 1/5 of 2/3 of a day basketing' (mud). (It is easy to check that 2/3 + 1/5 · 2/3 = 4/5, since 5 · (2/3 + 1/5 · 2/3) = 6 · 2/3 = 4.)

§2.2.18. The proposed explanation of the solution algorithm in *ZA* 94, 3, is simultaneously a possible explanation of the obscure passage in lines 3-5 of *ZA* 94, 3, where the phrase 'its 6th-part set, its 4th-part set' may mean 'take the 6th-part of 1 ninda, and (add) the 4th-part of that'.

§2.2.19. It is awkward that the proposed explanation of *ZA* 94, 3, hinges on the assumption that 2$_{aš}$ in line 1 of *ZA* 94, 3, means 2 · 100 rather than 2 · 60. It is also awkward that the area number recorded in line 2 of *ZA* 94, 3, suffers from what may be a fatal flaw, namely the assumption that 2$_{diš}$ aša$_5$ can stand for 2/3 iku. The fractions of the iku that are known to occur in other cuneiform texts are 1/2, 1/4, and 1/8 iku. In addition, the only known examples of the occurrence of 2$_{diš}$ as a sign for 2/3 are the ones mentioned above, where in all cases 2$_{diš}$ šar stands for 2/3 šar. Therefore, maybe it would be wise to follow Foster and Robson in assuming that the area number given in line 2 of *ZA* 94, 3, is meant to be a fraction of the šar rather than of the iku. In that case, the author of this text, probably a student who listened to his teacher's oral instructions, made the double mistake of writing aša$_5$ instead of šar and of using a sign for 1/4 iku as a sign for 1/4 šar.

§2.2.20. Thus, a slightly different alternative interpretation of *ZA* 94, 3, is that the length number 2$_{aš}$ in line 1 of the text simply stands for 2 <ninda>, not 200$^!$ <ninda>, and that the curiously written area number in line 2 stands for 2/3 šar$^!$ - 1/4 šar$^!$, not 2/3 iku - 1/4 iku. In that case, the first step of the solution algorithm proposed above can be changed to the more straightforward

1. A/2 = (2/3 šar - 1/4 šar)/2 = (40 - 15) area-shekels/2
 = 12 1/2 area-shekels.

The remaining three steps of the proposed solution algorithm are not affected by the change.

§3. *Three earlier published Old Akkadian metric division exercises*

§3.1. In *ZA* 94, 3, Foster and Robson briefly mention three earlier published Old Akkadian metric division exercises and claim that in all three of them the answer was found by use of sexagesimal numbers in place value notation. This is almost certainly not correct. The alternative discussion below of the three metric division exercises in question is borrowed (with some amendments) from Friberg nd, app. 6, A6 c. The hand copies in figures 3-4 are based on copies of the texts in *DPA* and *TMH* 5.

§3.2. *DPA 38 and TMH 5, 65, Old Akkadian metric division exercises of standard type (figure 3)*

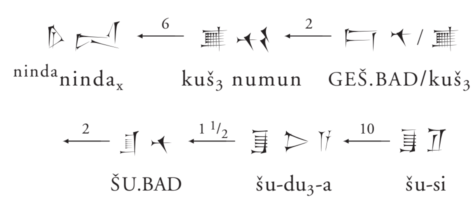
Note that *DPA* 38 (=PUL 29) is erroneously called PUL 54 in *ZA* 94, 3.

§3.2.1. This is the brief text of *DPA* 38 (according to Limet probably from Girsu/Lagash):

2$_{\text{geš2}}$ 4$_{\text{u}}$ uš ḫi	*2 geš$_2$ 40 the average length.*
sag 1$_{\text{iku}}$ aša$_5$	*The front, 1 iku.*
sag-bi 3 kuš$_3$ numun	*Its front 3 seed-cubits*
1 GEŠ.BAD 1 ŠU.BAD	*1 GEŠ.BAD 1 ŠU.BAD*

§3.2.2. In order to understand this and the following two exercises, it is necessary to be familiar with the following nearly complete version of the factor diagram for Old Akkadian ninda fractions:

L(OAkk) :



§3.2.3. In *DPA* 38 the front s (the short side of a rectangle) has to be computed when the length u = 2 geš$_2$ 40 (ninda) and the area A = 1 iku. The answer is given in the form

s = 3 seed-cubits 1 GEŠ.BAD 1 ŠU.BAD (= *3 1/2 1/4 seed-cubits*).

§3.2.4. It is clear that here 2 geš$_2$ 40 (just like 9 geš$_2$ in CMAA 016-C0005) is a regular sexagesimal number, since 2 geš$_2$ 40 = 160 = 32 · 5 = 2^5 · 5. However, before the invention of place value notation and abstract numbers the front cannot have been computed as (the number for) the area times the reciprocal of (the number for) the length. Instead, the front may have been computed by use of a factorization algorithm like the one proposed above in the case of CMAA 016-C0005. In the case of *DPA* 38, the factorization algorithm can have taken the following form:

1 iku	=	10 n. · 10 n.	*a square with the side 10 ninda has the area 1 iku*
	=	40 n. · 2 n. 3 s.c.	*one side multiplied by 4, the other by 1/4*
	=	2 geš$_2$ 40 n. · 3 1/2 1/4 s.c.	*the length multiplied by 4, the front by 1/4*

Hence, the answer is that the front is 3 1/2 1/4 s.c. = 3 seed-cubits 1 GEŠ.BAD 1 ŠU.BAD.

§3.3.1. *TMH* 5, 65 (Westenholz 1975, no. 65) is a similar text, probably from Nippur. The length u = 1 geš$_2$ 7 1/2 ninda, the area A is again = 1 iku, and the answer is given in the form

s = 1 $^{\text{ninda}}$ninda$_x$(DU) 5 kuš$_3$ 2 šu-du$_3$-a 3 šu-si 1/3 šu-si (= *1 ninda 5 2/3 cubits 3 1/3 fingers*)

§3.3.2. The ninda fractions appearing in this text are not quite the same as those in *DPA* 38. However, the given length is here, again, a regular sexagesimal number (times 1 ninda), since 1 geš$_2$ 7 1/2 = 27 · 2 1/2 = 27/4 · 10. The solution can be obtained by use of the following factorization algorithm:

1 iku	=	10 n. · 10 n.	*a square with the side 10 ninda has the area 1 iku*

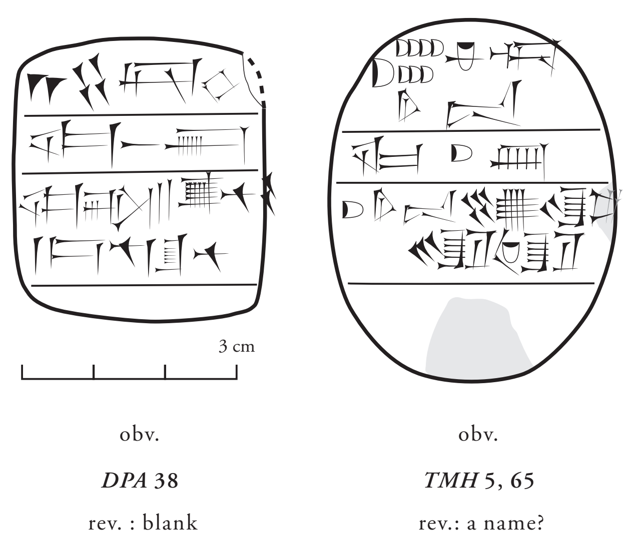
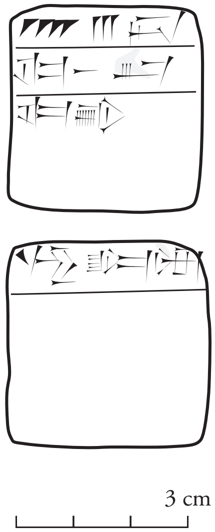


Fig. 3. DPA 38 and TMH 5, 65, two Old Akkadian metric division exercises.



4$_{\text{geš}2}$ 3 uš
sag 1$_{\text{iku}}$ aša$_5$
sag-bi

pa$_3$-de$_3$-dam
'to be found'

Fig. 4. DPA 39, an Old Akkadian mathematical assignment with a metric division problem.

=	30 n. · 3 n. 4 c.	*one side multiplied by 3, the other by 1/3*
=	45 n. · 2 n. 2 c. 2 šu-du$_3$-a	*the length multiplied by 1 1/2, the front by 2/3*
=	1 geš$_2$ 7 1/2 n. · 1 n. 5 2/3 c. 3 1/3 f.	*the length multiplied by 1 1/2, the front by 2/3*

Hence, the answer is that the front is 1 n. 5 2/3 c. 3 1/3 f. = 1 ninda 5 kuš$_3$ 2 šu-du$_3$-a 3 1/3 šu-si.

§3.3.3. *DPA 39, an Old Akkadian metric division exercise without answer (figure 4)*

Note that *DPA* 39 (PUL 31) is erroneously called PUL 23 in *ZA* 94, 3.

§3.3.4. *DPA* 39 is a fourth text of the same type as CMAA 016-C0005, *DPA* 38, and *TMH* 5, 65, except that no answer is given in this text. Instead a blank and the phrase pa$_3$-de$_3$-dam 'to be found' follow the phrase sag-bi 'its front (is)'. The given area is again A = 1 iku, and the given length is u = 4 geš$_2$ 3 ninda. Here 4 geš$_2$ 3 is again a regular sexagesimal number, since 4 geš$_2$ 3 = 3 · 1 21 = 3^5. The teacher's intention may once again have been that the answer should be obtained by use of a factorization algorithm.

§3.3.5. An added difficulty in this case is that the answer cannot be expressed in presently known Old Akkadian units of length. However, assume, for the sake of the argument, that the small fractional length unit 1 še 'barley-corn' = 1/6 finger was in use already in the Old Akkadian period, although there is no textual evidence to support this assumption (cf. Powell 1990, §I.2, and Friberg 1993, texts 4, 5, 9, and 12) If this assumption is admitted, then the factorization algorithm in this case may have been as follows:

1 iku	=	10 n. · 10 n.	
	=	12 n. · 8 n. 4 c.	*one side multiplied by 1 1/5, the other by 1 - 1/6*
	=	18 n. · 5 1/3 n. 2/3 c.	*the length multiplied by 1 1/2, the front by 2/3*
	=	27 n. · 3 1/2 n. 2 1/3 c. 3 1/3 f.	*the length multiplied by 1 1/2, the front by 2/3*
	=	1 geš$_2$ 21 n. · 1 n. 2 2/3 c. 4 1/3 f. 2/3 b.c.	*the length multiplied by 3, the front by 1/3*
	=	4 geš$_2$ 3 n. · 4 2/3 c. 8 f. 2/3 b.c. and 1/3 of 2/3 b.c.	*the length multiplied by 3, the front by 1/3*

Thus the answer in this case would be

s = 4 2/3 cubits 8 fingers 2/3 barley-corn and 1/3 of 2/3 barley-corn.

§3.3.6. However, this answer would not have fitted into the small space available on the clay tablet! That may be the reason why no answer is recorded there.

Cf. Friberg nd, fig. 1.4.5 # 2, where a student had been asked to compute the reciprocal of 1 30 48 06 02 15 25, a 7-place regular sexagesimal number. Expecting the answer to be another 7-place number he did not make enough room for the answer on his clay tablet. The answer, however turned out to be the 18-place number 39 38 48 38 28 37 02 08 43 27 09 43 15 53 05 11 06 40, which the student had to record as a number beginning with one line on the obverse, continuing around the edge, and ending with two lines on the reverse.

§3.3.7. It is remarkable that the three length numbers 2 geš$_2$ 40 n. = (1 + 1/3) · 2 geš$_2$, 1 geš$_2$ 7 1/2 n. = (1 + 1/8) · 1 geš$_2$, and 4 geš$_2$ 3 n. = (1 + 1/80) · 4 geš$_2$ in *DPA* 38, *TMH* 5, 65, and *DPA* 39 all are regular sexagesimal numbers (times 1 ninda). These are the earliest known attestations of (deliberately chosen) regular sexagesimal numbers, foreshadowing the enormous interest attached to such numbers in both Old and Late Babylonian mathematical texts. This point was missed by Foster and Robson, who even suggested (2004, footnote 9) that "perhaps 4 03 is a scribal error for 4 30."

§4. ***On true and alleged antecedents of Old Babylonian sexagesimal place value notation***

§4.1. *Sexagesimal place value numbers in texts from the Neo-Sumerian Ur III period*

§4.1.1. M. Powell was the first person to seriously study mathematical cuneiform texts from the 3rd millennium BC. In *Historia Mathematica* 3 (1976) pp. 417-439, he wrote about "The antecedents of Old Babylonian place notation and the early history of Babylonian mathematics." He correctly noted that sexagesimal place value notation was used in *YOS* 4, 293, a (non-mathematical) Ur III text, which can be dated, on the basis of a year formula ('the year <Enunugalanna was installed as> en-priest of Inanna <in Uruk>') to the fifth year of Amar-Suen, 2043 BC. Therefore, it seems to be definitely established that place value notation for sexagesimal numbers was already in use before the end of the 3rd Dynasty of Ur.

§4.1.2. Powell commented the physical appearance of *YOS* 4, 293 (see fig. 5) in the following way:

> *But this document has a greater significance than is evident from the cuneiform copy. When I collated this text in the Yale Babylonian Collection (June 17, 1974), it turned out to be a kind of ancient 'scratch pad'. It has a form similar to a school text, being rather thick and having flat edges. The writing surface is extremely flat, and the back side, which is not used, is convex. The writing surface shows clear traces of having been previously used. The appearance of the tablet suggests that it was moistened and smoothed off after use. Here we have at last an explanation of why so little trace of sexagesimal notation has survived from the Ur III period. [Here Powell adds a footnote mentioning other examples he knows about.] Calculations in sexagesimal notation were made on temporary tablets which were then moistened and erased for reuse after the calculation had been transferred to an archival document in standard notation.*

§4.1.3. Two columns of sexagesimal numbers in place value notation are recorded on the obverse of *YOS* 4, 293, one (here called A) neatly in the first text case, the other one (here called B), in some disorder after the apparent end of the inscription. The numbers in columns A and B are the following:

A	14 54	B	2 54
	29 56 50		45
	17 43 40		28
	30 53 20		17
			2 28
			27

The sums of the numbers in the two columns are not recorded, but clearly they are, respectively,

sum A = 1 33 27 40, and
sum B = 7 19.

These two totals have to be interpreted as

sum A = 1 33;27 40 shekels, and
sum B = 7 19 shekels.

§4.1.4. The totals can be converted to ordinary Sumerian/Babylonian weight numbers as follows:

1 33;27 40 shekels = 1 $^1/_2$ mina 3 $^1/_2$ shekel - 7 barley-corns (1 barley-corn = $^1/_{180}$ shekel = ;00 20 shekel)

and

7 19 shekels = 7 minas 19 shekels.

§4.1.5. As shown in the "conform transliteration" of *YOS* 4, 293, in figure 5, these totals are actually recorded on the clay tablet. Sum A is specified as mu-ku$_x$(DU) didli, probably meaning 'diverse deliveries',

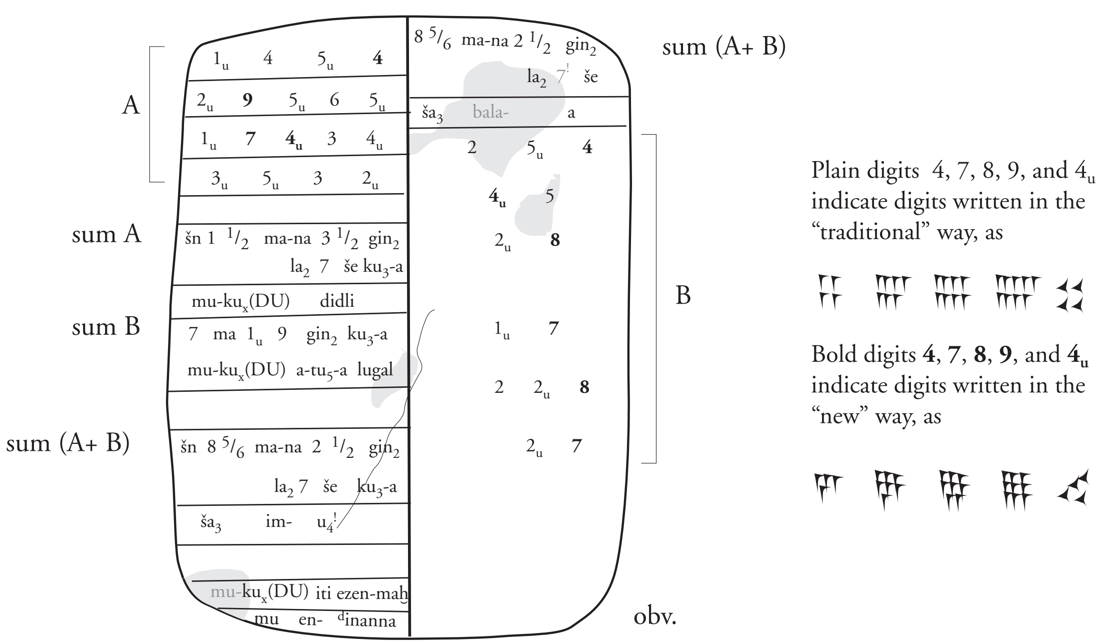


Fig. 5. YOS 4, 293, an Ur III text with sexagesimal numbers in place value notation

and sum B as mu-ku$_x$(DU) a-tu$_5$-a lugal ‘deliveries for the ritual cleaning of the king’. In addition, the grand total is also recorded (twice), that is the sum of these two totals:

sum (A + B) = sum A + sum B
= 1 33;27 40 shekels + 7 19 shekels
= 8 52;27 40 shekels
= 8 5/6 minas 2 1/2 shekels
- 7 barely-corns.

§4.1.6. The first grand total is specified as ša$_3$ im-u$_4$ ‘from the daily tablets’, the second grand total (possibly) as ša$_3$ [bala]-a ‘obligatory payments’.

§4.1.7. The numbers in column A are the place value notation equivalents of the following numbers:

A	14 54	shekels	=	14 5/6 minas 4 shekels,
	29 56;50	shekels	=	29 5/6 minas 6 5/6 shekels,
	17 43;40	shekels	=	17 2/3 minas 3 2/3 shekels,
	30 53;20	shekels	=	30 5/6 minas 3 1/3 shekels.

Similarly, the numbers in column B can be written in standard notation for weight numbers as

B	2 54	shekels	=	2 5/6 minas 4 shekels,
	45	shekels	=	2/3 mina 5 shekels,
	28	shekels	=	1/3 mina 8 shekels,
	17	shekels	=	17 shekels,
	2 28	shekels	=	2 1/3 minas 8 shekels,
	27	shekels	=	1/3 mina 7 shekels.

§4.1.8. In *YOS* 4, 293, there are two ways of writing the digits 4, 7, 8, 9, and 40, either in the “traditional”, Sumerian way with horizontally extended number signs, or in the “new” way (the one most often used in Old Babylonian cuneiform texts) with vertically extended signs. In some Old Babylonian mathematical texts, such as the famous table text *MCT* 38 (Plimpton 322), the new way of writing digits is used for place value numbers in tables and computations, while the traditional way of writing digits is used in ordinary numbers, such as line numbers or traditional metrological numbers. The same tendency is clear in *YOS* 4, 293, where the traditional way of writing digits is used in the totals and grand totals, while the new way is used, although not quite consistently, in the place value numbers. It is likely that the more compact vertical form was invented in order to make it easier to fit the number signs of long sexagesimal numbers into narrow columns, optimally with ones and tens in orderly columns as in the first text case of *YOS* 4, 293. (Surprisingly, both ways of writing 4 are used in line 1 of *YOS* 4, 293, and both ways of writing 40 in line 3.) Note in this connection Powell’s remark (1976, 421)

> *Moreover, another unusual characteristic of this text suggests how the Sumerian system of (place value) notation functioned without a sign for zero: the sexagesimal numerals are arranged (in A and B) in quite clear columns according to their proper power.*

§4.1.9. So far, the only known example of a mathematical cuneiform text from the Ur III period using sexagesimal numbers in place value notation is the small brick metrology exercise *RTC* 413 (see Friberg, 1990, fig. 3).

§4.1.10. It is difficult to imagine that sexagesimal numbers in place value notation could be widely used without recourse to sexagesimal multiplication tables and tables of reciprocals. Although arithmetic table texts inscribed almost exclusively with numbers are hard to date, there are several reasons to believe that a small group of distinctly atypical tables of reciprocals are from Ur III. The group in question includes the Nippur text *MKT* 1, 10 (HS 201; s. Oelsner, *ChV* [2001]), the Nippur text Ist. Ni. 374 (Proust 2004,

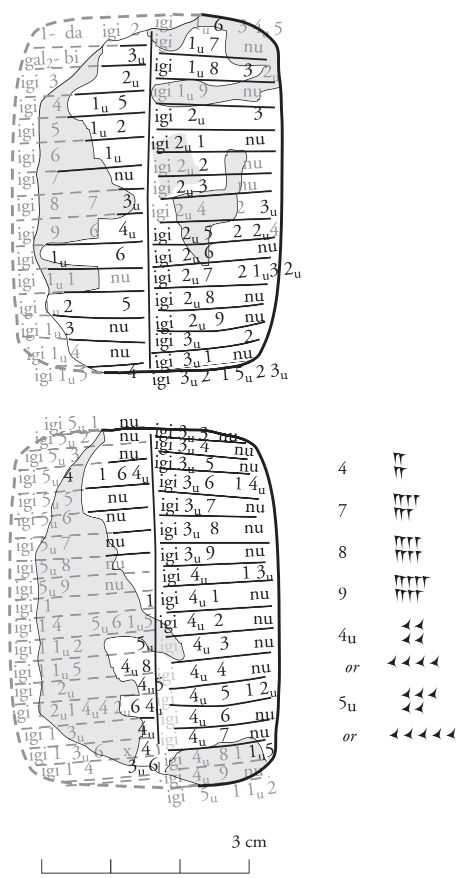


Fig. 6. Ist. Ni. 374, an Ur III table of reciprocals

ch. 6.2.2 and Table 1), and the Tello/Girsu text *ITT* 4, 7535 (s. Friberg nd, app. 1, fig. A1.2).

According to Proust 2004, 118, four more texts of the same type have been found by B. Lafont in Istanbul, and two more, both from Umma, have been found by E. Robson at the British Museum.

§4.1.11. The conform transliteration of Ist. Ni. 374 in figure 6 is based on a hand copy in Proust 2004. Note the similarity with *YOS* 4, 293: both tablets are of the same form and are inscribed in two columns. The traditional forms of the digits 4, 7, 8, 9, and 40 are used everywhere in Ist. Ni. 374. However, also here there seems to be a certain difference between the forms of the digits used for the ordinary numbers n and those used for the sexagesimal place value numbers recorded in the form igi n. In the former, the digits 40 and (probably) [50] are written with one line of tens, while in the latter, 40 and 50 are written more compactly with two lines of tens.

§4.2. *The alleged use of sexagesimal place value numbers in Old Akkadian texts.*

§4.2.1. After discussing *YOS* 4, 293, Powell went on to the Old Akkadian mathematical text *DPA* 38 (see above, fig. 3, left), suggesting that the metric division problem in that text was solved as follows:

$$\begin{aligned} {}^{1\ 40}/_{2\ 40} &= 1\ 40 \cdot {}^{1}/_{2\ 40} \\ &= 1\ 40 \cdot 0;00\ 22\ 30 \\ &= 0;37\ 30, \\ s = {}^{A}/_{u} &= {}^{1\ 40\ \text{sq. ninda}}/_{2\ 40\ \text{ninda}} \\ &= 0;37\ 30\ \text{ninda} \\ &= 3\ \text{seed-cubits}\ 1\ \text{cubit}\ 1\ \text{half-cubit}. \end{aligned}$$

§4.2.2. In the same vein, he suggested that another metric division problem in the Old Akkadian mathematical text *DPA* 39 (above, fig. 4) was solved as follows:

$$\begin{aligned} s = {}^{A}/_{u} &= {}^{1\ 40}/_{4\ 03} \\ &= 1\ 40 \cdot {}^{1}/_{4\ 03} \\ &= 1\ 40 \cdot 0;00\ 14\ 48\ 53\ 20 \\ &= 0;24\ 41\ 28\ 53\ 20\ \text{(ninda)} = ? \end{aligned}$$

§4.2.3. Also the computation producing the answer in *TMH* 5, 65 (above, fig. 3, right) was explained by Powell in a similar way, namely as

$$\begin{aligned} s = {}^{A}/_{u} &= {}^{1\ 40}/_{1\ 07;30} \\ &= 1\ 40 \cdot {}^{1}/_{1\ 07;30} \\ &= 1\ 40 \cdot 0;00\ 53\ 20 \\ &= 1;28\ 53\ 20\ \text{(ninda)} \\ &= 1\ \text{n.}\ 5\ {}^{2}/_{3}\ \text{c.}\ 3\ {}^{1}/_{3}\ \text{f.} \end{aligned}$$

§4.2.4. Powell's suggested explanations were repeated, without further discussion of the matter, in three "summaries of calculation" in Foster and Robson, *ZA* 94, 3. However, as was shown above, the metric division problems in the three mentioned Old Akkadian texts can easily have been solved by use of simple factorization algorithms, and there is no evidence for calculations with sexagesimal place value numbers in these or any other Old Akkadian mathematical texts.

§4.2.5. Also following in Powell's footsteps, Whiting wrote in *ZA* 74 (1984) pp. 59-66 a paper with the title "More Evidence for Sexagesimal Calculations in the Third Millennium B.C." In that paper, he discussed the three Old Akkadian square-side-and-area exercises *DPA* 36-37, and *ZA* 74, p. 60 (A 5443). He thought he could find evidence for counting with sexagesimal numbers in place value notation in all three of them. However, a close look at Whiting's arguments will show that they are based on an inadequate understanding of the meaning of place value notation, and on a lacking familiarity with the peculiarities of pre-Babylonian mathematical cuneiform texts (cf. the detailed discussion of mathematical cuneiform texts from the 3rd millennium BC in Friberg, nd, ch. 6 and app. 6). The three square-side-and-area exercises mentioned by Whiting will be discussed again below. The vector graphic copies of the author in figures 7, 9, and 11 are based on the copies of the texts in Limet 1973 and Whiting 1984.

§4.3. *DPA 36, an Old Akkadian square-side-and-area exercise (figure 7)*

§4.3.1. Take, for instance, *DPA* 36. In that text, the area of a square with the side

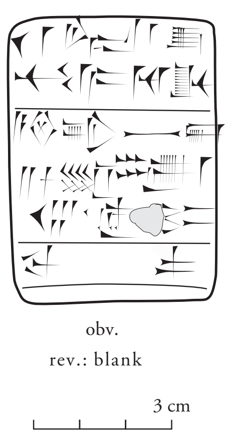


Fig. 7. DPA 36, an Old Akkadian square-side-and-area exercise.

s = 11 ninda 1 kuš$_2$ numun 1 GEŠ.BAD 1 ŠU.BAD

is given as

A = 1$_{iku}$ $^1/_{4}$$_{iku}$ aša$_5$ 2 $^1/_2$ šar 6 gin$_2$ 15 gin$_2$ tur.

§4.3.2. Whiting expresses both the side and the area of the square in *DPA* 36 in terms of sexagesimal numbers in place value notation, stating that

> *The side of the square is given as 11;17, 30 nindan and the correct area is 2,7;30,6,15 sar. The answer given in the tablet is 1 $^1/_4$ (iku) GANA$_2$ 2 $^1/_2$ sar 6 gin$_2$ 15 gin$_2$-tur which Powell interprets as 2,7;36,15. However, the answer given by the student need not be considered an error if we allow the positional use of the gin$_2$ sign. In this case we would interpret $^1/_2$ sar 6 gin$_2$ 15 gin$_2$-tur as ;30,6,15 which is the correct answer. Note that this is the same problem that would arise if sexagesimal place notation were used since 30,6 and 36 are both written with exactly the same characters in this system.*

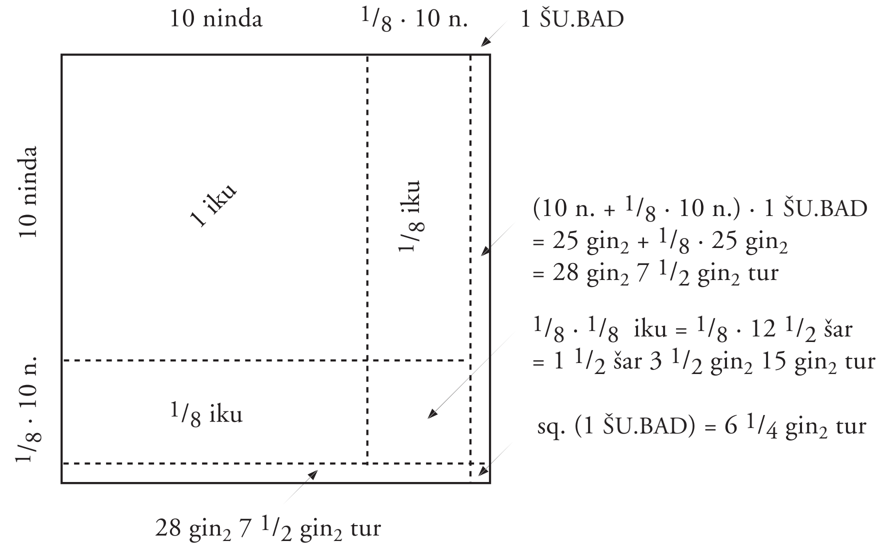


Fig. 8. DPA 36. The area of the square of a fractionally and marginally expanded length number.

§4.3.3. Thus, both Powell and Whiting assume that the area of the square was computed by first converting the given length number to a sexagesimal number in place value notation, then squaring this number, and finally converting the resulting sexagesimal number back to an area number. In the process, a simple positional error resulted in writing 6 gin$_2$ 15 gin$_2$ tur (= 6 $^1/_4$ gin$_2$) instead of 6 $^1/_4$ gin$_2$ tur.

§4.3.4. Contrary to the mentioned assumptions by Powell and Whiting, there is really no need to postulate that the author of *DPA* 36 counted with sexagesimal numbers in place value notation. Instead, he can have started by observing that since the kuš$_3$ numun 'seed-cubit' = $^1/_6$ ninda, and since the GEŠ.BAD = $^1/_2$ seed-cubit and the ŠU.BAD = $^1/_4$ seed-cubit, the given length of the side of the square can be interpreted as a "fractionally and marginally expanded length number". Indeed, it can be expressed in the following form:

s = 11 n. 1 $^1/_2$ k.n. 1 ŠU.BAD
= 11 $^1/_4$ n. 1 ŠU.BAD
= 10 n. + $^1/_8$ of 10 n. + 1 ŠU.BAD.

Here 10 ninda is a round length number, which is first expanded by a fraction of its length ($^1/_8$), and then further expanded with a small added segment (1 ŠU.BAD = $^1/_{24}$ ninda = $^1/_{240}$ of 10 ninda).

§4.3.5. The (area of the) square of this length number can have been be computed in a few simple steps, with the first step reasonably being to compute the square of only the first two terms:

sq. (10 n. + $^1/_8$ · 10 n.)
= sq. 10 n. + 2 · $^1/_8$ · sq. 10 n. + sq. $^1/_8$ · sq. 10 n.
= 1 iku + 2 · $^1/_8$ iku + $^1/_8$ · $^1/_8$ iku
= 1 $^1/_4$ iku + $^1/_8$ · 12 $^1/_2$ šar
= 1 $^1/_4$ iku 1 $^1/_2$ šar 3 $^1/_2$ gin$_2$ 15 gin$_2$ tur.

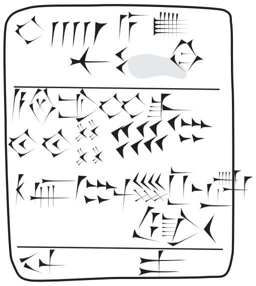
Geometrically, a computation like this can be explained as an application of the easily observed "square expansion rule" that the square of a length composed of two unequal parts is a square composed of four parts, namely two unequal squares and two equal rectangles. See figure 8.

§4.3.6. In the second step of the computation, a second application of the same rule shows that

sq. s = sq. (10 n. + $^1/_8$ of 10 n. + $^1/_4$ of $^1/_6$ n.)
= sq. (10 n. + $^1/_8$ · 10 n.) + 2 · (10 n. + $^1/_8$ of 10 n.) · 1 ŠU.BAD + sq. (1 ŠU.BAD)
= 1 $^1/_4$ iku 1 $^1/_2$ šar 3 $^1/_2$ gin$_2$ 15 gin$_2$ tur + 2 · (25 gin$_2$ + $^1/_8$ · 25 gin$_2$) + 6 $^1/_2$ gin$_2$ tur
= 1 $^1/_4$ iku 2 $^1/_2$ šar 6 $^1/_4$ gin$_2$ tur.

Note that, since 1 ŠU.BAD = $^1/_4$ seed-cubit = $^1/_4$ · $^1/_6$ ninda, it follows that

1 ninda · 1 ŠU.BAD = $^1/_4$ · $^1/_6$ sq. ninda
= $^1/_4$ · $^1/_6$ šar = $^1/_4$ · 10 gin$_2$
= 2 $^1/_2$ gin$_2$,
sq. (1 ŠU.BAD) = $^1/_4$ · $^1/_6$ · 2 $^1/_2$ gin$_2$



1$_{\text{šar2}}$ 5$_{\text{geš2}}$ la$_2$ 1 kuš$_3$
numun si$_8$
a-ša$_3$-bi 2$_{\text{šar2}}$-gal
2$_{\text{šar'u}}$ 4$_{\text{bur'u}}$ 9$_{\text{bur3}}$ 5$_{\text{iku}}$
$^1/_{8}$$_{\text{iku}}$ aša$_5$ 5$_{\text{aš}}$ $^1/_2$ šar 1$_{\text{aš}}$ gin$_2$
$^2/_3$-ša
ba-pa

obv.
rev. blank

3 cm

Fig. 9. DPA 37, another Old Akkadian square-side-and-area exercise

= $^1/_4$ · 25 gin$_2$ tur
= 6 $^1/_4$ gin$_2$ tur.

§4.3.7. The calculation of sq. s above was based on these relations. It is likely that they were supposed to be well known by Old Akkadian school boys, but that the author of *DPA* 36 did not correctly remember the value of sq. (1 ŠU.BAD), which he took to be 6 $^1/_4$ gin$_2$ instead of 6 $^1/_4$ gin$_2$ tur.

§4.4. *DPA 37, another Old Akkadian square-side-and-area exercise (figure 9)*

§4.4.1. Now consider the parallel text *DPA* 37, in which the side and the area of a square are

s = 1 šar$_2$ 5 geš$_2$ ninda - 1 seed-cubit,

and

A = 2 šar$_2$-gal 20 šar$_2$ 49 bur$_3$ 5 iku $^1/_8$ iku
5 $^1/_2$ šar 1 gin$_2$ $^2/_3$.

§4.4.2. Whiting 1984 interpreted the given length and area numbers in this text as 1 04 59;50 ninda and 1 10 24 38 38; 01 40 šar, respectively, but noted that the correct value for the area would have been 1 10 24 38 20; 01 40 šar. He explained the error as a dittography, 38 38 instead of 38 20, a "very common type of error in place notation".

§4.4.3. However, I am unable to cite any other example of a dittography of this type in a mathematical cuneiform text, and just as in the case of *DPA* 36, there is no need to postulate that the author of *DPA* 37 counted with sexagesimal numbers in place value notation. Note that in *DPA* 37 the given side can be characterized as a "fractionally expanded and marginally contracted length number". Indeed,

s = 1 šar$_2$ 5 geš$_2$ ninda - 1 seed-cubit
= 1 šar$_2$ ninda + $^1/_{12}$ of 1 šar$_2$ ninda - 1 seed-cubit.

Note also that here the subtracted seed-cubit is really marginal, since

1 šar$_2$ 5 geš$_2$ ninda - 1 seed-cubit
= 3,900 ninda - $^1/_6$ ninda
= appr. 23.4 kilometers - 1 meter.

§4.4.4. The square of this length number can have been computed in a few easy steps, more or less in the same way as the square of the corresponding length number in *DPA* 36. Indeed, it is likely that the computation was based on an application of a combination of the square expansion rule mentioned above with the similar "square contraction rule" that the square of the difference of two unequal lengths is equal to two unequal squares minus two equal rectangles. As shown in figure 10, the first step of the computation ought to have been the computation of the square of the first two terms of the given length number:

sq. (1 šar$_2$ 5 geš$_2$ ninda)
= sq. (1 šar$_2$ n. + $^1/_{12}$ ·
1 šar$_2$ n.)
= sq. 1 šar$_2$ n.
+ 2 · $^1/_{12}$ · sq. 1 šar$_2$ n.
+ sq. $^1/_{12}$ · sq. 1 šar$_2$ n.

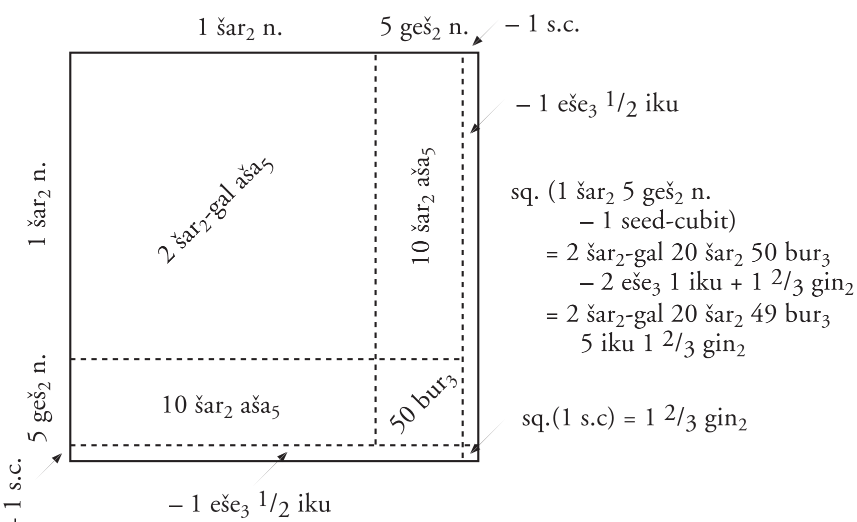
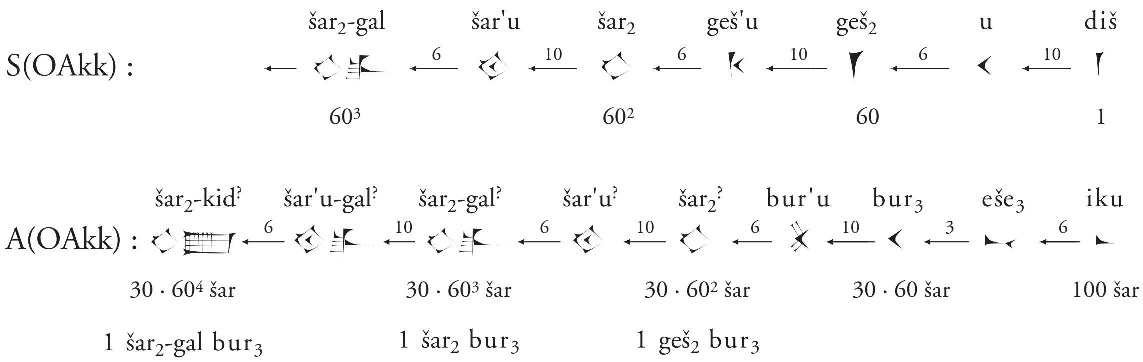


Fig. 10. DPA 37. The area of the square of a fractionally expanded and marginally contracted length number.

= 2 šar$_2$-gal + 2 · 10 šar$_2$
+ 50 bur$_3$.

§4.4.5. This computation can be explained by use of the factor diagrams below for sexagesimal numbers and large area measures, respectively.



These are the cuneiform variants of the number signs, as opposed to the round, curviform variants.

§4.4.6. In the second step of the computation in *DPA* 37, an application of the square contraction rule shows that

sq. (1 šar$_2$ 5 geš$_2$ n. - 1 s.c.)
= sq. (1 šar$_2$ 5 geš$_2$ ninda) - 2 · (1 eše$_2$ $1/2$ iku)
+ sq. 1 s.c.

This computation can be explained by use of the following multiplication table for length measures:

1 s.c. · 1 s.c.	= $1/6 \cdot 1/6$ šar	(*s.c. = seed-cubit*)
	= $1/6$ · 10 gin$_2$	
	= 1 $2/3$ gin$_2$,	
1 n. · 1 s.c.	= $1/6$ šar	
	= 10 gin$_2$,	
1 geš$_2$ n. · 1 s.c.	= 1 geš$_2$ · 10 gin$_2$	
	= 10 šar,	
1 šar$_2$ n. · 1 s.c.	= 1 geš$_2$ · 10 šar	
	= 1 eše$_2$.	

§4.4.7. In view of the computations above, the correct answer in the case of *DPA* 37 ought to be

sq. (1 šar$_2$ 5 geš$_2$ n. - 1 s.c.)
= 2 šar$_2$-gal 20 šar$_2$ 50 bur$_3$
- 2 eše$_3$ 1 iku + $1/6 \cdot 1/6$ šar
= 2 šar$_2$-gal 20 šar$_2$ 49 bur$_3$ 5 iku
1 $2/3$ gin$_2$.

However, the answer given in the text is slightly different, namely

sq. (1 šar$_2$ 5 geš$_2$ n. - 1 s.c.)
= 2 šar$_2$-gal 20 šar$_2$ 49 bur$_3$ 5 $1/8$ iku
5 $1/2$ šar 1 $2/3$ gin$_2$.

The difference between the correct and the recorded answer is

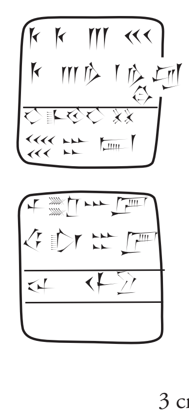
$1/8$ iku 5 $1/2$ šar = 12 $1/2$ šar + 5 $1/2$ šar = 18 šar.

§4.4.8. This curious error can possibly be explained as follows: The student who wrote the text may have became confused by the 1 eše$_3$ $1/2$ iku resulting from an intermediate step of the computation, and counted in the following incorrect way:

sq. (1 šar$_2$ 5 geš$_2$ n. - 1 s.c.)
= 2 šar$_2$-gal 20 šar$_2$ 50 bur$_3$ - 2 · (1 eše$_3$ $1/2$ iku)
+ $1/6 \cdot 1/6$ · (1 eše$_3$ $1/2$ iku) - $1/6 \cdot 1/6$ šar
= 2 šar$_2$-gal 20 šar$_2$ 50 bur$_3$ - 2 eše$_3$ 1 iku
+ $1/6$ · 1 iku 8 $1/3$ šar - $1/6$ · 10 gin$_2$
= 2 šar$_2$-gal 20 šar$_2$ 49 bur$_3$ 5 iku
+ 18 šar 3 $1/3$ gin$_2$ - 1 $2/3$ gin$_2$
= 2 šar$_2$-gal 20 šar$_2$ 49 bur$_3$ 5 $1/8$ iku
5 $1/2$ šar 1 $2/3$ gin$_2$.

§4.5. *A 5443, an Old Akkadian square-side-and-area exercise with decimal numbers (figure 11)*

§4.5.1. The very small clay tablet *ZA* 74, p. 60, A 5443, is an Old Akkadian square-side-and-area



33$_{\text{geš2}}$ 33
ninda 1 nig$_2$-kas$_7$ si$_8$
1$_{\text{šar2}}$-gal 11$_{\text{šar2}}$ 27$_{\text{bur3}}$
5$_{\text{iku}}$ aša$_5$
$1/2$ šar 3$_{\text{aš}}$ gin$_2$
$2/3$-ša 5$_{\text{aš}}$ gin$_2$ <tur>
ba-pa$_3$

Fig. 11. ZA 74, p. 60, A 5443. An Old Akkadian square-side-and-area exercise with numbers that don't make sense.

exercise in which the numbers don't make sense. (The given length number is also strangely written with one of the signs for 1 geš'u = 10 geš$_2$ misplaced.) It is potentially important to note that both the length number and the area number in A 5443 have, partially, the form of "funny numbers". Thus, the given length number contains the digit 3 written 5 times, and the area number begins with the digit 1 repeated three times (1 šar$_2$-gal 11 šar$_2$; cf. *DPA* 36 in fig. 7 above, where the given length number is written with the digit 1 repeated five times.) Actually, since 1 nikkas = $1/4$ ninda = 3 cubits, the given length number can be expressed as a number with the digit 3 repeated five times, namely as 33 geš$_2$ 33 ninda 3 cubits!

§4.5.2. The area of a square with the side 33 geš$_2$ ninda is 36 šar$_2$ 18 bur$_3$. Therefore, a square with the side 33 geš$_2$ 33 ninda 3 cubits, as in A 5443, cannot have the indicated area,

A = 1 šar$_2$-gal 11 šar$_2$ 27 bur$_3$ 5 iku $1/2$ šar 3 $2/3$ gin$_2$ 5 gin$_2$ <tur>.

A possible partial explanation of the strange numbers in A 5443 is that they may be an example of experimentation with the traditional Sumerian number systems, in an attempt to make them decimal.

Known examples of such experiments are two Old Babylonian mathematical texts from Mari. See Chambon, 2002, Proust, 2002, and Soubeyran 1984.

§4.5.3. Therefore, suppose that in A 5443, contrary to the normal situation,

1 geš'u = 10 geš$_2$ = 1,000
1 geš$_2$ = 100.

Then 33 geš$_2$ 33 ninda 1 nikkas in this text means 3,333 $1/4$ ninda, and

sq. 3,333 n. 3 c.
= (sq. 3,333. + $1/2 \cdot$ 3,333 + $1/16$) sq. n.
= (11,108,889 + 1,666 $1/2$ + $1/16$) sq. n.
= 11,110,555 $1/2$ $1/16$ sq. n.
= 111,105 iku 55 $1/2$ šar 3 $2/3$ gin$_2$ 5 gin$_2$ tur.

§4.5.4. Comparing this result with the area number recorded on A 5443, one is led to the tentative conclusion that in this text the large units of area measure may have had the following decimal values:

1 šar$_2$-gal	= 100,000 iku
1 šar'u	= 10,000 iku
1 šar$_2$	= 1,000 iku,
1 bur'u	= 100 iku,
1 bur$_3$	= 10 iku.

Under these assumptions, the area number recorded in A 5443 can be interpreted as meaning

A = 111,275 iku $1/2$ šar$_2$ 3 $2/3$ gin$_2$ 5 gin$_2$ <tur>.

§4.5.5. Clearly, several of the initial and final digits in the computed area number are the same as the corresponding digits in the recorded area number. Unfortunately, the digits in the middle are not the same in the computed and the recorded area numbers. Presumably, the lack of agreement is due to some counting error, but an attempt to establish the precise nature of that error has not been successful. However, at least the last part of the recorded area number is correct, since

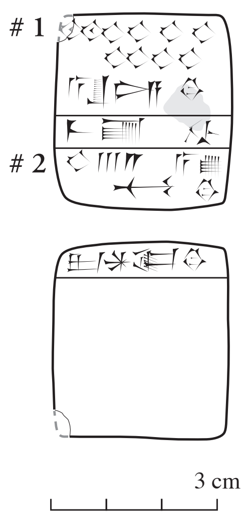
sq. 1 nikkas = $1/16$ šar = 3 $2/3$ gin$_2$ 5 gin$_2$ tur.

It is interesting to note, by the way, that the given length number in A 5443 can be interpreted as

3,333 ninda 3 cubits = 3,333 $1/3$ ninda - 1 cubit.

This would be an explanation for the four initial digits 1 in sq. 3,333 n. 3 c., since

sq. 3,333 $1/3$ ninda = sq. ($1/3 \cdot$ 10,000 n.)
= $1/9 \cdot$ 100,000,000 sq. n.
= 11,111,111 $1/9$ sq. n.



2$_{\text{šar'u}}$ 8$_{\text{šar2}}$
la$_2$ 1 šu-du$_3$-a si$_8$
me-lu-ḫa
1$_{\text{šar2}}$ 5$_{\text{geš2}}$ la$_2$ 1 kuš$_3$
numun si$_8$
ur-$^{\text{d}}$ištaran

Fig. 12. A 5446. Two Old Akkadian assignments of square-side-and-area exercises.

§4.6. *ZA 74, p. 65, A 5446, two Old Akkadian square-side-and-area assignments without answer (figure 12)*

§4.6.1. *ZA* 74, p. 65, A 5446 is another Old Akkadian mathematical text published by Whiting 1984.

It appears to be a teacher's note to himself that he has handed out two square-side-and-area exercises as assignments to two named students, Meluḫḫa and Ur-Ištaran. One of the assignments (# 2) is the one to which the answer is given in *DPA* 37 (§4.4 above). In the other assignment (# 1), the given length of the square-side is

28 šar$_2$ - 1 šu-du$_3$-a = 30 šar$_2$ - $^1/_{15}$ of 30 šar$_2$
- 1 šu-du$_3$-a.

§4.6.2. The area of the square in A 5446 # 1 can have been computed in a few simple steps, just like the areas of the squares in *DPA* 36-37. The first step would probably have been to compute

sq. (30 šar$_2$ - $^1/_{15}$ · 30 šar$_2$)
= 30 šar$_2$-kid - 2 · 2 šar$_2$-kid + 8 šar$_2$-gal
= 26 šar$_2$-kid 8 šar$_2$-gal.

§4.6.3. For the second step of the computation, it would have been necessary to know the following relations:

1 šu-du$_3$-a = $^1/_6$ seed-cubit
= $^1/_6$ · $^1/_6$ ninda,
1 n. · 1 šu-du$_3$-a = $^1/_6$ · $^1/_6$ šar
= $^1/_6$ · 10 gin$_2$
= 1 $^2/_3$ gin$_2$,
1 geš$_2$ n. · 1 šu-du$_3$-a = 1 geš$_2$ · 1 $^2/_3$ gin$_2$
= 1 $^2/_3$ šar,

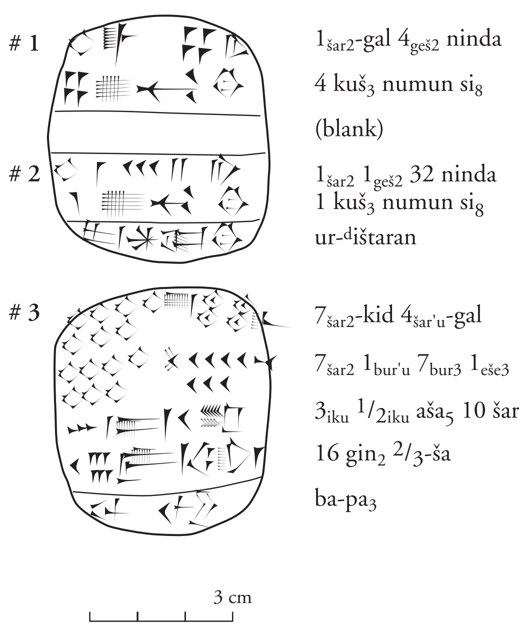


Fig. 13. MAD 5, 112. Three square-side-and-area exercises: two questions and one answer.

1 šar$_2$ n. · 1 šu-du$_3$-a = 1 geš$_2$ · 1 $^2/_3$ šar
= 100 šar = 1 iku,
1 šu-du$_3$-a · 1 šu-du$_3$-a = $^1/_6$ · $^1/_6$ · 1 $^2/_3$ gin$_2$
= 2 $^2/_3$ gin$_2$ tur and
$^1/_6$ of $^2/_3$ gin$_2$ tur.

§4.6.4. Clearly, then, the area of the square in A 5446 # 1 would have been found to be

26 šar$_2$-kid 8 šar$_2$-gal - 2 · 28 iku + 2 $^2/_3$ gin$_2$ tur
and $^1/_6$ of $^2/_3$ gin$_2$ tur
= 26 šar$_2$-kid 7 šar$_2$-gal 59 šar$_2$ 56 bur$_3$ 2 eše$_3$
4 iku 2 $^2/_3$ gin$_2$ tur and $^1/_6$ of $^2/_3$ gin$_2$ tur.

In Whiting 1984, the length of the given side of the square in A 5446 # 1 is expressed in sexagesimal numbers in place value notation as 27 59 59;58 20 ninda. Whiting wisely refrained from computing the square of this sexagesimal number and then trying to convert the result into Old Akkadian area numbers. It would have been more laborious than using the direct method indicated above.

§4.6.5. It should be noted that in the case of A 5446 # 1, the side of the square can be interpreted as a "fractionally and marginally contracted length number". The side of the square in *DPA* 36 (§4.3) is a "fractionally and marginally expanded length number", and the side of the square in *DPA* 37 is a "fractionally expanded and marginally contracted length number". This can hardly be a coincidence. Instead, it is clear that the square-side-and-area problems in A 5446 and *DPA* 36-37 are based on three closely related but different variants of a clever geometric construction. The interesting conclusion of this observation must be that some anonymous mathematician in the Old Akkadian period (c. 2340-2200 BC) knew how to make use of one of the most fundamental mathematical tools, namely the systematic variation of a basic idea.

§4.6.6. It is not only the switching between expansions and contractions that catches the eye in the three mentioned texts. It is also the use of three different small units of length as the marginally added or subtracted numbers, namely the ŠU.BAD = $^1/_4$ seed-cubit = $^1/_4$ · $^1/_6$ ninda in *DPA* 36, the seed-cubit = $^1/_6$ ninda in *DPA* 37, and the šu-du$_3$-a = $^1/_6$ seed-cubit = $^1/_6$ · $^1/_6$ ninda in A 5446. In addition to this, in A 5443 (section 3.5), a marginally added unit of length is the nikkas = $^1/_4$ ninda. It was clearly the Old Akkadian teachers' aim to teach their students how to handle all possible difficulties that could arise in the use of the Old Akkadian systems of length and area numbers.

§4.7. *MAD 5, 112, an Old Akkadian text with three assignments (figure 13)*

§4.7.1. The next text, *MAD* 5, 112 (Ashm. 1924.689), is inscribed on the obverse with two large length numbers, plus a name, Ur-Ištaran, perhaps the same Ur-Ištaran as the student who got an assignment on A 5446 and wrote his answer on *DPA* 37. On the reverse is recorded a very large area number. Powell 1976,p. 429, interpreted the text as giving the area of a very large rectangle, with the sides of the rectangle recorded on the obverse of the clay tablet, and wrote the following comment:

> *Ur-Ištaran was doubtless an unhappy little Sumerian when he recited the result of his computation which we find on the reverse of the tablet, for, the correct solution is 1, 1, 36, 16, 49, 41;26, 40 šar (or square nindan), but the number computed by the student apparently equals 3, 50, 0, 38, 46, 0;16, 40 šar.*

§4.7.2. Whiting 1984, on the other hand, realized that *MAD* 5, 112, could be a parallel to A 5446, and therefore interpreted the text on the obverse as two assignments, the first to an individual whose name has been left out, the other to Ur-Ištaran. However, he conceded that

> *The area given on the reverse does not help resolve the interpretation of the obverse since it represents neither the product of the two numbers nor the square of either of them.*

§4.7.3. Actually, as will be shown below, the area number on the reverse is (very close to) the square of a large length number (that it is not an exact square is probably due to a small computational error). Therefore, the correct interpretation of the text seems to be that there are two questions to square-side-and-area exercises on the obverse of *MAD* 5, 112, and one answer to an unrelated square-side-and-area exercise on the reverse.

§4.7.4. The given length number in assignment # 1 is bigger than all the length numbers in *DPA 36*-37 and A 5446:

1 šar$_2$-gal 4 geš$_2$ ninda 4 kuš$_3$ numun
= appr. 1,296 km + 1,440 m + 4 m.

The area of a square with this side would be immense. Note, in particular, that

sq. 1 šar$_2$-gal ninda
= $2 \cdot 60^4$ bur$_3$
= $2 \cdot 60^4 \cdot$ appr. 64,800 square meters
= appr. 1,680,000 sq. km.

The Sumerian/Old Akkadian designation for 60^4 bur$_3$ is not known.

§4.7.5. The length number in assignment # 1 is 1 šar$_2$-gal 4 geš$_2$ ninda 4 kuš$_3$ numun. It is composed of one large length number, one of intermediate size, and one that is quite small. It is likely that the teacher's intention was that the square expansion rule would be used twice for the computation of the area of a square with this side.

§4.7.6. The length number in assignment # 2 is 1 šar$_2$ 1 geš$_2$ 32 ninda 1 kuš$_3$ numun, which is again the sum of one large, one intermediate, and one small length number.

§4.7.7. The area number recorded in assignment # 3 on the reverse of *MAD* 5, 112, is

A = 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 17 bur$_3$ 1 eše$_3$ 3 $^1/_2$ iku aša$_5$ 10 šar 16 gin$_2$ $^2/_3$ <gin$_2$>.

This area number can be shown to be the answer to a square-side-and-area exercise. It is possible that the teacher's instruction to one of his students was to find the side of a square of this area. If that is so, then the

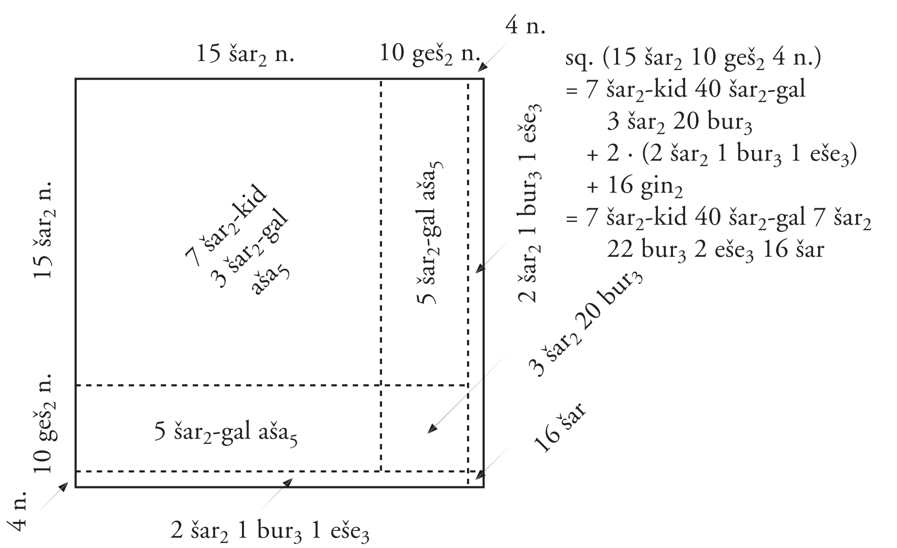


Fig. 14. MAD 5, 112, rev. The first three steps of the computation of the side of a given square.

method the student was assumed to use must have been a metro-mathematical variant of the "square side rule" often used in both Old and Late Babylonian mathematical texts to compute the sides of squares with given areas, or the square-roots of given non-square integers (cf. the detailed discussion in Friberg 1997, §8.)

§4.7.8. The idea behind the square side rule is simply an application, repeated as many times as necessary, of the square expansion rule in reverse. Therefore, the same kind of geometric model that can be used to explain the computation in several steps of the area of a square can also be used to explain the computation in several steps of the side of a square. Figure 14 demonstrates the first three steps of the computation of the side of the square with the area given in the assignment on the reverse of *MAD* 5, 112.

§4.7.9. Below are indicated the successive steps of the proposed computation of the square side.

A = 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 17 bur$_3$ 1 eše$_3$ 3 $^1/_2$ iku aša$_5$ 10 šar 16 gin$_2$ $^2/_3$ <gin$_2$>

1a. A = appr. 7 šar$_2$-kid 30 šar$_2$-gal aša$_5$ = 7 šar$_2$-gal 30 šar$_2$ bur$_3$ = 3 šar$_2$-gal 45 šar$_2$ · 1 šar$_2$ šar = sq. 15 šar$_2$ n.

1b. 1st square side appr. s_1 = 15 šar$_2$ n.

1c. Deficit D_1 = A - sq. s_1 = (appr.) 10 šar$_2$-gal aša$_5$

2a. D_1 / 2s_1 = 10 šar$_2$-gal aša$_5$ / 30 šar$_2$ n. = 10 šar$_2$ bur$_3$ / 30 šar$_2$ n. = 1 eše$_3$ / 1 n. = 10 geš$_2$ n.

2b. 2nd square side appr. s_2 = 15 šar$_2$ 10 geš$_2$ n., sq. s_2 = 7 šar$_2$-kid 40 šar$_2$-gal 3 šar$_2$ 20 bur$_3$ aša$_5$

2c. Deficit D_2 = A - sq. s_2 = (appr.) 4 šar$_2$ aša$_5$

3a. D_2 / 2s_2 = (appr.) 4 šar$_2$ aša$_5$ / 30 šar$_2$ n. = 4 geš$_2$ bur$_3$ / 30 šar$_2$ n. = 4 bur$_3$ / 30 geš$_2$ n. = 4 n.

3b. 3rd square side appr. s_3 = 15 šar$_2$ 10 geš$_2$ 4 n., sq. s_3 = 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 22 bur$_3$ 2 eše$_3$ 16 šar

3c. Excess E_3 = sq. s_3 - A = (appr.) 5 bur$_3$ (a negative deficit)

4a. E_3 / 2s_3 = (appr.) 5 bur$_3$ / 30 šar$_2$ n. = 5 šar / 1 geš$_2$ n. = $^1/_{12}$ n. = 1 kuš$_3$ (cubit)

4b. 4th square side appr. s_4 = 15 šar$_2$ 10 geš$_2$ 4 n. - 1 k. sq. s_4 = 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 22 bur$_3$ 2 eše$_3$ 16 šar - 5 bur$_3$ 1 iku $^2/_3$ šar + sq. (1 k.) = 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 17 bur$_3$ 1 eše$_3$ 5 iku aša$_5$ 15 $^1/_3$ šar $^1/_3$ gin$_2$ 5 gin$_2$ tur

4c. Excess E_4 = sq. s_4 - A = (appr.) 1 $^1/_2$ iku

5a. E_4 / 2s_4 = (appr.) 1 $^1/_2$ iku / 30 šar$_2$ n. = 5 šar / 1 šar$_2$ n. = 5 n. / 1 šar$_2$ = $^1/_2$ šu-si (finger)

5b. 5th square side appr. s_5 = 15 šar$_2$ 10 geš$_2$ 4 ninda - 1 kuš$_3$ $^1/_2$ šu-si sq. s_5 = (appr.) 7 šar$_2$-kid 40 šar$_2$-gal 7 šar$_2$ 17 bur$_3$ 1 eše$_3$ 3 $^1/_2$ iku aša$_5$ 13 $^1/_2$ šar 9 $^2/_3$ gin$_2$ 6 gin$_2$ tur

5c. Excess E_5 = sq. s_5 - A = (appr.) 3 $^1/_2$ šar

§4.7.10. Here the computation comes to an end. In the next step, one would have to subtract $^5/_6$ of $^1/_6$ of a barley-corn from the approximate square side s_5, and there would still not be an exact fit. Therefore, the recorded area number on the reverse of the text *MAD* 5, 112, is not a perfect square, although it is very close to one. Apparently, the scribe who wrote the text made a small error near the end of his computation of the square of the side s = 15 šar$_2$ 10 geš$_2$ 4 ninda - 1 kuš$_3$ $^1/_2$ šu-si. The precise cause of the error is difficult to establish.

In place value numbers s = 15 šar$_2$ 10 geš$_2$ 4 ninda - 1 kuš$_2$ $^1/_2$ šu-si = 15 10 04 n. - ;05 05 n. = 15 10 03;54 55 n., and sq. s = 3 50 03 38 46 03;39 45 50 25 šar. The area number recorded on the reverse of *MAD* 5, 112, corresponds, in sexagesimal place value numbers, to 3 50 03 38 46 00;10 16 40. Note that a number of this form cannot be an exact square of a sexagesimal number. Indeed, if a = b · 60 + ;10 16 40, then 64 · a = 64 · b · 60 + 3 42 is a sexagesimal number with the last place 42, and no exact square of a sexagesimal number can have the last place 42.

§4.7.11. An intriguing extra twist is the astonishing circumstance that the side s_2 = 15 šar$_2$ 10 geš$_2$ ninda is not just an arbitrarily chosen length number. Indeed, the number 15 šar$_2$ 10 geš$_2$ can be factorized as

15 šar$_2$ 10 geš$_2$ n. = 10 geš$_2$ · 1 geš$_2$ 31 n.
= 10 geš$_2$ · 13 · 7 n.
= (1 + $^1/_{12}$) · (1 + $^1/_6$) · 12 šar$_2$ ninda.

§4.7.12. This is certainly not a coincidence, since the factorization shows that 15 šar$_2$ 10 geš$_2$ is a good example of an "almost round number". By definition, an almost round number is a number which is close to a round number and simultaneously equal to another round number multiplied by one or two factors of the kind (1 + $^1/_n$), where n is a small regular sexagesimal integer (note that 15 šar$_2$ 10 geš$_2$ is very close to the round number 15 šar$_2$, since 10 geš$_2$ is only $^1/_{90}$ of 15 šar$_2$). It is easy to find examples of almost round area numbers in proto-cuneiform field-area and field-sides texts from Uruk around the end of the 4th millennium BC (see Friberg 1997/1998, fig. 2.1). It is also easy to find examples of almost round numbers in Old Babylonian mathematical texts. See, for instance, Friberg nd, ch. 1.2 (MS 2831, a series of five squaring exercises), and ch. 8.1.b (MS 2107 and *MCT* 44 [YBC 7290], two computations of the area of a trapezoid). Two further examples of almost round numbers in Old Akkadian computations of areas are *DPA* 3 and *OIP* 14,

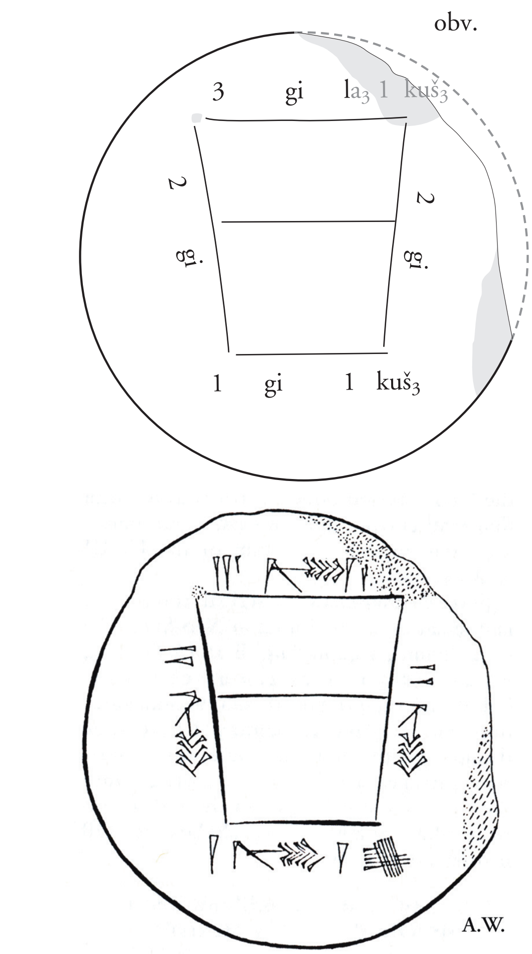


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Fig. 15. RlA 7, 541, IM 58045. An Old Akkadian trapezoid equipartition exercise.

§4.8. *RlA 7, p. 541, IM 58045, an Old Akkadian text with a drawing of a partitioned trapezoid (figure 15)*

§4.8.1. The Old Akkadian hand tablet IM 58045 (= 2N-T 600), a text from Nippur, is by its find site in a collapsed house firmly dated to the reign of the king Šarkališarri. There is drawn on it a trapezoid with a transversal line parallel to the fronts of the trapezoid. The lengths of all four sides of the trapezoid, but not the length of the transversal, are indicated in the drawing.

§4.8.2. The length of the trapezoid is given as 2 gi 'reeds' = 12 cubits (1 ninda). The lengths of the two fronts are m = 3 reeds - 1 [cubit] = 17 cubits and n = 1 reed 1 cubit = 7 cubits, respectively. It is known that in Old Babylonian mathematical texts, 17, 13, 7 is the most often occurring example of a "trapezoid triple", the term meaning that the area of a trapezoid with the fronts 17 and 7 is divided in two equal parts by a transversal of length 13 (see Friberg 1990, §5.4 k.).

§4.8.3. Presumably, the two notations '2 reeds' indicate not the long sides of the trapezoid, but the constant distance between the two parallel sides. Therefore, the area of the trapezoid can be computed as

$$\text{2 reeds} \cdot \text{((3 reeds - 1 cubit) + (1 reed 1 cubit))}/_2$$
$$= \text{2 reeds} \cdot \text{2 reeds} = \text{1 sq. ninda} = \text{1 šar,}$$

a conspicuously round number. It can have been the Old Akkadian teacher's intention that the length d of the transversal should be computed as follows (without any use of sexagesimal place value numbers):

$$\begin{aligned}\text{sq. d} &= \text{(sq. m + sq. n)}/_2\\ &= \text{((9 sq. reeds - 6 reed} \cdot \text{cubit + 1 sq. cubit)}\\ &\quad \text{+ (1 sq. reed + 2 reeds} \cdot \text{cubit + 1 sq. cubit))}/_2\\ &= \text{5 sq. reeds - 2 reeds} \cdot \text{cubit + 1 sq. cubit}\\ &= \text{1 šar 15 gin}_2\text{ - 5 gin}_2\text{ + }^1/_3\text{ gin}_2\text{ 5 gin}_2\text{ tur}\\ &= \text{1 šar 10 gin}_2\ ^1/_3\text{ gin}_2\text{ 5 gin}_2\text{ tur}\\ &= \text{sq. 2 reeds 1 cubit,}\end{aligned}$$

hence d = 2 reeds 1 cubit (= 13 cubits).

§4.8.4. Note that in the drawing on IM 58045 all lengths are given in the form of traditional length numbers. This is in contrast to drawings of trapezoids in Old Babylonian mathematical texts, where lengths normally are given in the form of abstract (sexagesimal) numbers. An Old Babylonian school boy would have computed the transversal of the trapezoid in figure 15 (essentially) as follows:

$$\begin{aligned}\text{sq. d} &= \text{(sq. m + sq. n)}/_2\\ &= \text{(sq. 1;25 + sq. ;35)}/_2\\ &= \text{(2;00 25 + ;20 25)}/_2\end{aligned}$$

116, both discussed in Friberg nd, app. 6.

A skeptical reader may object that every given number is very close to an exact square, namely the square of its approximate square root. However, that objection is not valid in the case of the area number recorded on the reverse of *MAD* 5, 112, since, as was shown above, that number is very close to the square of 15 šar$_2$ 10 geš$_2$ 4 ninda - 1 kuš$_3$ $^1/_2$ šu.si, which is a very special kind of length number. As remarked already, 15 šar$_2$ 10 geš$_2$ ninda is an almost round length number. In addition, just like the two length numbers recorded on the obverse of the clay tablet, the length number 15 šar$_2$ 10 geš$_2$ 4 ninda is composed of one large length number, one of intermediate size, and one that is quite small. The whole number 15 šar$_2$ 10 geš$_2$ 4 ninda - 1 kuš$_3$ $^1/_2$ šu.si can be understood as a two times expanded and two times contracted length number. Note also that the second term is 90 times smaller than the first term, the third term is 150 times smaller than the second term, the fourth term is 48 times smaller than the third term, and the fifth term is 60 times smaller than the fourth term.

= 1;10 25
d = sqs. 1;10 25
= 1;05 <ninda>.

§4.9. *OIP 14, 70, an ED IIIb table (Adab) of areas of small squares (figure 16)*

§4.9.1. Whiting 1984, p. 64, even thought that he could find evidence for counting with sexagesimal place value numbers in the table of small squares *OIP* 14, 70 (cf. Edzard 1969, 101-104), an ED IIIb text from Adab. Consequently, he ended his paper with the following statement:

> *In summary, the evidence provided by Powell supplemented by that presented here prompts me to state with confidence that sexagesimal place value notation was being used to perform calculations in the Old Akkadian period and that instruction in these techniques was being carried out at Lagash/Girsu and probably at Nippur. Less strong, but still significant, is the evidence that the use of sexagesimal place notation was known before the Sargonic period, especially for the expression of fractions.*

§4.9.2. The brief argument which Whiting uses in order to prove that sexagesimal place value notation was used by the author of *OIP* 14, 70, goes as follows:

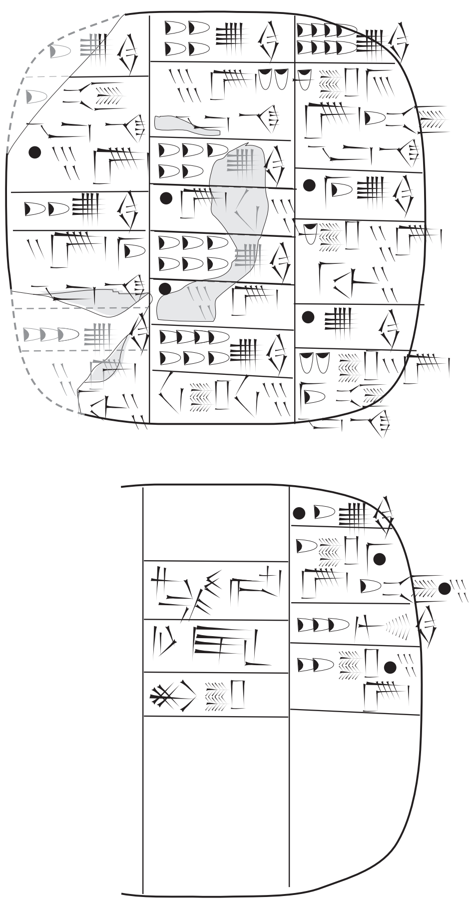


Fig. 16. OIP 14, 70. A table of areas of small squares from ED IIIb Adab.

> *These two lines (lines 10 and 14) give the squares of 5 cubits and 7 cubits respectively: (in sexagesimal notation) (;25 nindan)^2 = ;10 25 sar and (;35 nindan)^2 = ;20 25 sar. It can be seen that the quantities written in these lines, 10 gin$_2$ $^1/_3$ 6 and $^1/_3$ sar $^1/_3$ 5, exactly express the correct answers in sexagesimal place notation.*

§4.9.3. The argument is, of course, not compelling. It only expresses the trivial truth that the correct values of the squares, expressed in sexagesimal place value numbers, are equal to the correct values of the same squares expressed in Old Akkadian fractional notations.

§4.9.4. *OIP* 14, 70, is extensively discussed in Friberg nd, app. 1. It is shown there how all the values in the table of squares can have been computed with departure from the result recorded in the first line of the table, namely (in terms of exchange-minas and shekels of exchange-minas)

sq. 1 cubit = 1 $^1/_4$ sa$_{10}$ ma-na
= 1 sa$_{10}$ ma-na 15 <sa$_{10}$> gin$_2$.

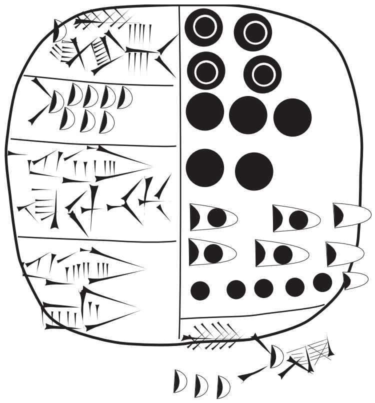
§4.9.5. That result, in its turn, can have been obtained as follows:

1 reed	= $^1/_2$ ninda
1 cubit	= $^1/_6$ reed,
1 sq. reed	= $^1/_4$ šar
	= 15 gin$_2$,
1 reed · 1 cubit	= 15 gin$_2$/6
	= 2 $^1/_2$ gin$_2$,
1 sq. cubit	= 2 $^1/_2$ gin$_2$/6
	= 7 $^1/_2$ sa$_{10}$ ma-na/6
	= 1 $^1/_4$ sa$_{10}$ ma-na
	= 1 sa$_{10}$ ma-na 15 <sa$_{10}$> gin$_2$.

§4.10. *TSŠ 50 and 671, metric division exercises from Šuruppak (ED IIIa; figure 17)*

§4.10.1. There are two known Old Sumerian division exercises from Šuruppak in the Early Dynastic IIIa period, about the middle of the 3rd millennium BC (see Friberg 1990, §4.2). One of them is *TSŠ* 50, where the question is how many men can receive rations of 7 sila$_3$ each from the barley in a granary. The answer is given, correctly, in non-positional sexagesimal numbers. The details of the solution procedure are not provided.

§4.10.2. A related text, also from Šuruppak, is *TSŠ* 671. Both *TSŠ* 50 and 671 were first published in Jestin 1937. Photos of the clay tablets can be found in Høyrup 1982.



obv.
rev. blank

$1_{\text{aš@c}}$ še gur_7	*One granary of barley.*
sila_3 $7_{\text{aš@c}}$	*7 sila_3*
$1_{\text{aš}}$ lu_2 šu ba-ti	*a single man receives.*
lu_2-bi	*Its men?*
$4_{\text{šar'u@c}}$ $5_{\text{šar2@c}}$	*45 šar_2 42 geš_2 51,*
$4_{\text{geš'u@c}}$ $2_{\text{geš2@c}}$	
$5_{\text{u@c}}$ $1_{\text{aš@c}}$	
še $3_{\text{aš@c}}$ sila_3 šu tag_4	*3 sila_3 of barley remains.*

Fig. 17. TSŠ 50, a metric division exercise from Šuruppak, mentioning a 'granary' (copy based on J. Høyrup's photo).

§4.10.3. *TSŠ* 50 and 671 are both concerned with the same division exercise. The unstated question can be reconstructed as follows:

> All the barley contained in 1 'granary' is to be divided into rations, so that each man (lu_2) or worker (guruš) receives 7 sila_3. How many men get their rations?

§4.10.4. The exact answer given in *TSŠ* 50 is

> 45 šar_2 42 geš_2 51 rations, with 3 sila_3 left over (šu tag_4).

Counting backwards, one finds that the barley contained in 1 'granary' must have been

> 7 sila_3 · 45 šar_2 42 geš_2 51 + 3 sila_3
> = 5 šar_2-gal 20 šar_2 sila_3.

§4.10.5. This result agrees fairly well with the known fact that in later cuneiform texts, both Sumerian and Babylonian, gur_7 'granary' was the name of a very large capacity unit, equal to 1 šar_2 gur. The gur was another large capacity unit, equal to 5 geš_2 (300) sila_3 in Old Babylonian mathematical texts, but of varying size in Sumerian administrative texts, depending on from what site and from which period the texts originate. The size of the sila_3 was also varying but was probably always roughly equal to 1 liter.

§4.10.6. What *TSŠ* 50 seems to tell us is that in Šuruppak, in the middle of the 3rd millennium BC, 1 gur_7 was equal to 1 šar_2 gur, with each gur equal to 5 geš_2 20 (320) sila_2. Actually, however, only two kinds of gur are documented in texts from Šuruppak, in some texts the gur maḫ, 'mighty gur', equal to 8 geš_2 sila_3, in other texts the lid_2-ga, equal to 4 geš_2 sila_3. The reason for the curious appearance in *TSŠ* 50 of what appears to be a gur of 5 geš_2 20 sila_3 is unknown, but it can possibly be explained as follows. (cf. Friberg 1997, §§7b, 7e, and fig. 7.1).

§4.10.7. In Old Babylonian mathematical problem texts and mathematical tables of constants, sila_3 measures of various sizes appear, explicitly or implicitly. The size of a sila_3 of a given kind is indicated by its "storing number" (*našpakum*), which can be explained as the number of sila_3 of that particular kind contained in a volume unit. One common type of sila_3 was the "cubic sila_3", a cube of side 6 fingers (;01 ninda), with the storage number "5" (meaning 5 00 sila_3 per volume-shekel). Another type of sila_3, known only from a table of constants, was the sila_3 of the 'granary' (*kārum*) with the storing number 7 30 (7 30 sila_3 per volume-shekel). Clearly the size of the granary sila_3 must have been only 2/3 of the size of an ordinary cubic sila_3, since 5 00 = 2/3 of 7 30. Therefore, 1 gur_7 of granary sila_3 would be as much as only 5 geš_2 20 cubic sila_3, the number of sila_3 counted with in *TSŠ* 50 and 671. This strange counting with two different storing numbers can be explained if it is assumed that, for some reason, contents of granaries were habitually counted in granary sila_3 while rations were counted in cubic sila_3 (unfortunately, nothing is known about why there existed sila_3-measures of several different sizes).

§4.10.8. Powell thought that the answers in *TSŠ* 50 and 671 had been computed by use of sexagesimal numbers in place value notation. Thus, in *HistMath* 3 (1976), 433, he suggests that the correct answer in *TSŠ* 50 might have been obtained by the following process, using an accurate reciprocal of 7:

(1)	5 20 00 00 · ;08 34 17 08	= 45 42 51;22 40,
(2)	45 42 51 · 7	= 5 19 59 57,
(3)	5 20 00 00 - 5 19 59 57	= 3.

Here the sexagesimal fraction ;08 34 17 08 is a fairly accurate approximation to the reciprocal of 7, since

> 7 · ;08 34 17 08 = ;59 59 59 56 = 1 - ;00 00 00 04.

§4.10.9. Powell also thought that the incorrect answer in *TSŠ* 671 (45 šar$_2$ 36 geš$_2$) could be explained as the result of using the slightly incorrect value ;08 33 for the reciprocal of 7. Indeed,

5 20 00 00 · ;08 33 = 5 20 · 8 33
= 42 45 00 + 2 51 00
= 45 36 00.

§4.10.10. On the other hand, Melville 2002, 237-252, was able to give a much more convincing explanation of how the answers in *TSŠ* 50 and 671 can have been computed by an ED IIIa school boy without recourse to sexagesimal numbers in place value notation. Although Melville failed to realize it, the solution algorithm he proposes is closely related to a solution algorithm in terms of non-positional decimal numbers used in *MEE* 3, 74 (TM.75.G 1392), a metric division exercise from Ebla, nearly as old as the texts from Šuruppak (Friberg 1986). Somewhat simplified and refined, Melville's explanation goes as follows in the case of *TSŠ* 50:

barley	*men receiving rations*	*remainder*
1) 1 ban$_2$ = 10 sila$_3$	1	3 sila$_3$
2) 1 barig = 6 ban$_2$	8	4 sila$_3$
3) 1 gur maḫ = 8 barig	1 geš$_2$ 8	4 sila$_3$
4) 10 gur maḫ	11 geš$_2$ 25	5 sila$_3$
5) 1 geš$_2$ gur maḫ	1 šar$_2$ 8 geš$_2$ 34	2 sila$_3$
6) 10 geš$_2$ gur maḫ	11 šar$_2$ 25 geš$_2$ 42	6 sila$_3$
7) 40 geš$_2$ gur maḫ	45 šar$_2$ 42 geš$_2$ 51	3 sila$_3$

the corresponding remainder calculations:

1)	*10 sila$_3$*	=	*1 ration + 3 sila$_3$*
2)	*6 · 3 sila$_3$*	=	*2 rations + 4 sila$_3$*
3)	*8 · 4 sila$_3$*	=	*4 rations + 4 sila$_3$*
4)	*10 · 4 sila$_3$*	=	*5 rations + 5 sila$_3$*
5)	*6 · 5 sila$_3$*	=	*4 rations + 2 sila$_3$*
6)	*10 · 2 sila$_3$*	=	*2 rations + 6 sila$_3$*
7)	*4 · 6 sila$_3$*	=	*3 rations + 3 sila$_3$*

§4.10.11. In the case of *TSŠ* 671, the author of the text apparently made a fatal mistake halfway through the solution algorithm, which was then propagated to the remaining steps of the solution procedure:

barley	*men receiving rations*	*remainder*
1) 1 ban$_2$ = 10 sila$_3$	1	3 sila$_3$
2) 1 barig = 6 ban$_2$	8	4 sila$_3$
3) 1 gur maḫ = 8 barig	1 geš$_2$ 8	4 sila$_3$
4) 10 gur maḫ	11 geš$_2$ 24	——
5) 1 geš$_2$ gur maḫ	1 šar$_2$ 8 geš$_2$ 24	——
6) 10 geš$_2$ gur maḫ	11 šar$_2$ 24	——
7) 40 geš$_2$ gur maḫ	45 šar$_2$ 36 geš$_2$	——

the corresponding remainder calculations:

1)	*10 sila$_3$*	=	*1 ration + 3 sila$_3$*
2)	*6 · 3 sila$_3$*	=	*2 rations + 4 sila$_3$*
3)	*8 · 4 sila$_3$*	=	*4 rations + 4 sila$_3$*
4)	*10 · 4 sila$_3$*	=	*4 ban$_2$* **= 4 rations!**

§4.10.12. The reason for the mistake may be that the normal size of a ration was 1 ban$_2$ (10 sila$_3$), and that the author of *TSŠ* 671 for a moment forgot that in this exercise it was supposed to be instead only 7 sila$_3$.

§5. *Conclusion*

§5.1. The detailed discussion above makes it clear that all computations in known mathematical cuneiform texts from the third millennium can be explained without the use of sexagesimal numbers in place value notation. Besides, explanations in terms of sexagesimal place value numbers severely obscure the fine points of the texts. It is like cracking nuts with a sledge hammer.

§5.2. The discussion in §§2-3 of Old Akkadian metric division texts shows that Old Akkadian mathematicians were familiar with the notion of regular sexagesimal numbers and suggests that they were familiar with the powerful idea of factorization algorithms. Similarly, the discussion in §§4.2-6 of Old Akkadian square-side-and-area texts suggests that Old Akkadian mathematicians were familiar with both funny numbers and almost round numbers, with the square expansion and square contraction rules, and with the powerful tool of a systematic variation of a basic idea.

§5.3. The discussion of *MAD* 5, 112, rev. in section §§4.7.7-11 reveals that Old Akkadian mathematicians might have been familiar with an efficient algorithm for the computation of the side of a square of given area, and the discussion of IM 58045 in section §4.8 shows that they were familiar also with the interesting topic of equipartitioned trapezoids.

§5.4. Finally, the discussion of ED IIIa metric division problems in §4.10 suggests that efficient metric division algorithms may have been known already in the ED III period.

§5.5. What is particularly important in this connection is that the mentioned crucial concepts and methods (with the possible exception of the metric division algorithms) also played important roles in Old Babylonian mathematics. Therefore, there can be little doubt that Old Babylonian mathematics had inherited many of its central ideas from its predecessors in the 3rd millennium BC.

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