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On Systems of the Second Degree — Some Thoughts (and a Lot of Pictures)

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Abstract

This paper is about a general method underlying the solution procedures for certain systems of equations of the second degree. It is deduced from the cuneiform examples of the elementary types of such systems and is then applied to the more sophisticated ones. The method also works for rather complex types which have not yet been found in the cuneiform record.

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0 Introduction

Among the great variety of problems the oB mathematical record presents us with — some solved, others only stated — are those that deal with systems of equations. I.e. with a set of two or more equations that have to be solved simultaneously. Often these systems are "of the second order" which means that some of the unknown magnitudes (which nowadays we like to denote by $x, y, z, \ldots$) occur as squares (like $x^2, y^2, z^2, \ldots$) or multiplied in pairs ($xy, yz, xz, \ldots$) or both. The subject of the present paper are certain subtypes of such systems that will be specified below.

Jens Høyrup discovered that these problems, while usually considered algebraic due to modern training, are in fact of a geometric nature in their oB setting. He found that many of their solution procedures originate in certain "cut-and-paste" operations which, however, are never explicitely described or even mentioned in the texts themselves. Høyrup issued a whole lot of publications on his findings but the most comprehensive and compelling one is his outstanding monograph Høyrup (2002) which, in this paper, will therefore be the only reference to his work. The following builds on these geometric ideas.

The types of systems considered here consist of one quadratic and one or more linear equations each. They are the following:

	Quadratic Equation		Linear Equations
I	$\sum_{i=1}^{n} x_i^2 = A$	a	$x_i = \frac{s_i}{t_i} x_{i-1} \quad (2 \le i \le n)$
		b	$x_i = x_{i-1} + b_i \quad (2 \le i \le n)$
		b$^{\mathrm{s}}$	$x_i = x_{i-1} + y \quad (2 \le i \le n)$ $\sum_{i=1}^{n} x_i = \Lambda$
		c	$x_i = \frac{s_i}{t_i} x_{i-1} + b_i \quad (2 \le i \le n)$
II	$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \gamma_i x_i = A$		
III	$\sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j + \sum_{i=1}^{n} \gamma_i x_i = A$	d	$x_i = \sum_{j=1}^{i-1} \frac{s_{ij}}{t_{ij}} x_j + b_i \quad (2 \le i \le n)$

The types Ia and Ib are what one might call "basic types" in that one fundamental idea each is used to solve them. It turns out that type Ic, on the other hand, as well as its generalizations, types IIc and IIId, can be solved by a subtle combination of these two ideas. Type Ib$^{\mathrm{s}}$ doesn't fit into this series of generalizations but is in fact a completely different generalization of Ib alone. This is indicated by the "s" superscript which doesn't stand for anything in particular, but you may think of "special" or "super" if you like.

The types are presented in the order of increasing complexity: Ia, Ib, Ic, IIc, IIId. Due to its special nature, type Ib^{s} follows directly after Ib. The interdependences are as follows:

$$
\begin{array}{ccccc}
\mathrm{Ia} & & \mathrm{Ib} & & \\
 & \searrow \quad \swarrow & & \searrow & \\
 & \mathrm{Ic} & & & \mathrm{Ib}^{\mathrm{s}} \\
 & \downarrow & & & \\
 & \mathrm{IIc} & & & \\
 & \downarrow & & & \\
 & \mathrm{IIId} & & &
\end{array}
$$

The expositions for types Ia and Ib start with explicit examples from the cuneiform record and give a formal account of the general case later. For type Ic this order is reversed in order to illuminate distinctly how the solution procedure can be seen as a combination of those for Ia and Ib. Although the types IIc and IIId are rather straight forward generalizations of Ic, the formal description of their general cases is more complicated. But since so far these two types lack, as to my knowledge, for any cuneiform examples anyway, there is no bothering about whether to do the examples first or the general case.[1]

The purpose of this exposition is twofold. First, to demonstrate the large variety of systems (among them quite complex ones) that can be solved applying nothing but two basic modes of operation the Mesopotamian mathematicians have used in the much fewer and simpler types that have come to us on clay. This does not mean to speculate whether they have really solved them, but just to indicate that the methods developed by them are sufficient to cope with the task. The second purpose is to present the solutions to the problems (be they present in the cuneiform record or not) as clearly as possible which I'm convinced is best done with a lot of pictures. In fact, it is my hope that at least for the less complicated cuneiform examples — that would be the types Ia, Ib, Ib^{s}, the examples in the appendices, and maybe even the two variable version of type Ic — the present exposition might serve as a textbook-like introduction for the mathematically non-preoccupied assyriologist, student and working professional alike. It is also for this reason that I have decided to avoid anything like Friberg's and Høyrup's "conformal translations" and special notations. While these are most valuable and provide a lot of insight to the enlightened, in my classroom experience they have proven an unnecessary obstacle to the novice. I also aim at making it possible for the reader to pick single examples and understand them without detailed knowledge of the earlier ones. This may render the exposition somewhat redundant at times. The illustrations of the text examples are not drawn to scale, i.e. do not represent the exact (ratios of) sizes given in the text. This is in order to highlight the general idea of the construction as well as to avoid odd shapes.

Readers who aren't fond of large amounts of formal notation or even mathematical formulae in general may skip the sections titeled "The general case" without serious danger of losing track of the principal ideas. At least I hope so. Even in section 5 (Type IIc) where there is nothing but the general case you may get a pretty good idea by ignoring the maths and just read the text and study the pictures. The same holds, in principle, for section 6. Only here you have to imagine the pictures — or draw them yourself.

[1] Actually, e.g. YBC 4714 no. 6 is of type IIc but gives no solution procedure, see Neugebauer (1935, 499).

A word about notation: Due to the special features of the Babylonian sexagesimal number notation (there is no sign for the digit “zero” and no sign for the “sexagesimal point”) the numbers are not uniquely determined by their cuneiform representation. E.g., a cuneiform “20” can denote $20 \cdot 60^n$ for all integer numbers n, i.e. ... , $20 \cdot 60^2$, $20 \cdot 60$, 20 , $\frac{20}{60}$, ... For the sake of convenience and legibility it is customary to *interpret* the numbers (i.e. to select one out of the many possibilities) in the translation (not in the transliteration, though). This is done by introducing a sign for the digit zero (here 00) and a sign for the sexagesimal point (here ▲) and making a choice for the position of the sexagesimal point (i.e. for the positional values of the digits). Keep in mind that this choice is arbitrary; in the examples given below I have not attempted to achieve any kind of consistency in this regard (except for inside each example, of course). Changing the interpretation amounts to moving the sexagesimal point. Be aware that (like in our decimal notation) if one moves the sexagesimal point in a number by n steps one has to move it by $2n$ steps in the number that represents the first number’s square (and the other way round for the square root). And one has to move it by n steps *in the other direction* in the number that represents the first number’s inverse.

1 The Type Ia

In this type of problem the givens are the sum of the areas of a number of unknown squares, and the respective ratios of their subsequent side lengths. For example, in the case of two unknown squares this can be formalized as

$$x^2 + y^2 = A \quad , \quad y = \frac{s}{t}x$$

where x and y are the unknown side lengths, A is the given combined area of the two squares and $\frac{s}{t}$ is the given ratio of their side lengths.

1.1 Two Variables

BM 13901, no. 10 (obv. ii, 11-18)[2] (after Neugebauer (1937, 2-3))

11) **a-ša$_3$** *ši-ta mi-it-ha-ra-ti-ia ak-mur-ma* 21 15
12) *mi-it-har-tum a-na mi-it-har-tim se-bi-a-tim im-ṭi*
13) 07 *u$_3$* 06 *ta-la-pa-at* 07 *u$_3$* 07 *tu-uš-ta-kal* 49
14) 06 *u$_3$* 06 *tu-uš-ta-kal* 36 *u$_3$* 49 *ta-ka-mar-ma*
15) 01 25 **igi**-01 25 *u$_2$-la ip-pa-ṭa-ar mi-nam a-na* 01 25
16) *lu-uš-ku-un ša* 21 15 *i-na-di-nam* 15-**e** 30 **ib$_2$-sa$_2$**
17) 30 *a-na* 07 *ta-na-ši-ma* 03 30 *mi-it-har-tum iš-ti-a-at*
18) 30 *a-na* 06 *ta-na-ši-ma* 03 *mi-it-har-tum ša-ni-tum*

(A) I have added the areas of my two squares and 21 15 00 (is the result).
(B) (The one) square side is (with respect to the other) square side reduced by (the factor of) one seventh.
(C) You write down 07 and 06.
(D_1) You multiply 07 and 07 (and) 49 (is the result).
(D_2) You multiply 06 and 06.
(E) You add (the resulting) 36 and (the) 49 (from step D_1) and 01 25 (is the result).
(F) The inverse of 01 25 cannot be solved.
(G) What shall I multiply by 01 25 that gives (the) 21 15 00 (from step A)? (The answer is 15 00.)
(H) The square root of (this) 15 00 is 30.
(I_1) You multiply (this) 30 by (the) 07 (written down in step C) and (the resulting) 03 30 is the first square side.
(I_2) You multiply (the same) 30 by (the) 06 (written down in step C) and (the resulting) 03 00 is the second square side.

Remark 1.1 Here and in some examples that follow one finds the phrase **igi**-*n ula ippaṭṭar* "the inverse of n cannot be solved". This means that $\frac{1}{n}$ cannot be written as a finite sexagesimal fraction. (This is because n contains prime factors other than 2, 3, and 5 which are the prime factors of the basis 60.) Using modern mathematical terminology one could

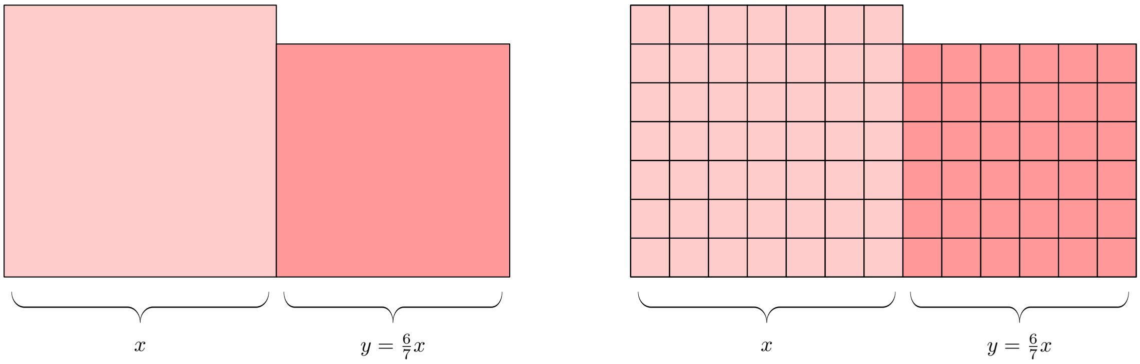
[2] See Høyrup (2002, 58-60).

therefore translate *paṭāru* as "to expand". The answer to the subsequent phrase "What shall I multiply by n that gives A" yields $\frac{A}{n}$ since $\mathbf{?} \cdot n = A \Rightarrow \mathbf{?} = \frac{A}{n}$.

In modern notation the given problem is to solve the following set of equations where x and y denote the unknown side lengths of "my two squares". The first equation translates step (A) whereas the second equation translates step (B).

$$x^2 + y^2 = 21\ 15\ 00 \quad , \quad y = \frac{6}{7}x$$

The solution procedure starts by recording the integers that represent the ratio of the unknown side lengths, namely 7 and 6 (step C): if one subdivides the side of the larger square into 7 equal parts, 6 of these fit exactly into the side of the smaller square. This subdivision of the square sides into 7 and 6 equal pieces, respectively, results in the subdivision of the squares themselves into $7^2 = 49$ and $6^2 = 36$ equal little squares (of also unknown side length):



In the steps (D_1) and (D_2) these multiplications are performed, and in step (E) their results are added ($36 + 49 = 01\ 25$) thus obtaining the total number of little squares. The combined area of all these 01 25 little squares is of course equal to the combined area of the two unknown squares whis is given as 21 15 00 (in step A). So in order to find the area of each of the little squares one has to divide 21 15 00 by 01 25. This is done by means of steps (F) and (G), see Remark 1.1. The result, namely 15 00, is therefore the area of one little square. Extracting the square root from this (step H) gives the side length of the little squares: $\sqrt{15 \cdot 60} = 30$. Taking this side length of the little squares 07 respectively 06 times (steps I_1 and I_2) one obtains the side lengths of the two previously unknown squares because they are built up from 07 respectively 06 little pieces.

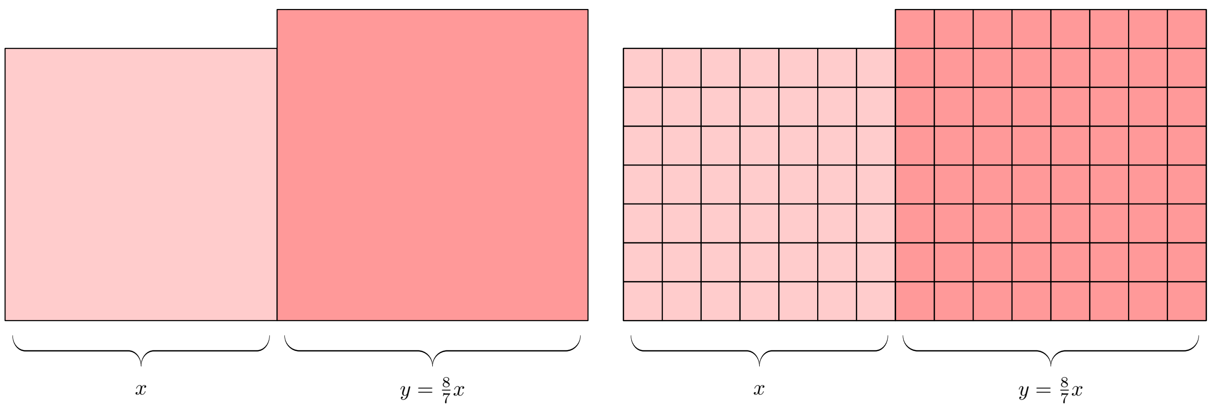
The next example is completely analogous to the one above, except that here the ratio between the two unknown side lengths is different (namely $\frac{8}{7}$ instead of $\frac{6}{7}$). The single steps of the solution procedure translate one-to-one just by replacing the numbers.

BM 13901, no. 11 (obv. ii, 19-26) (after Neugebauer (1937, 3))

19) **a-ša$_3$** *ši-ta mi-it-ha-ra-ti-ia ak-mur-ma* 28 15
20) *mi-it-har-tum* **ugu** *mi-it-har-tim se-bi-a-tim i-te-er*
21) 08 *u$_3$* 07 *ta-la-pa-at* 08 *u$_3$* 08 *tu-uš-ta-kal* 01 04
22) 07 *u$_3$* 07 *tu-uš-ta-kal* 49 *u$_3$* 01 04 *ta-ka-mar* 01 53
23) **igi**-01 53 *u$_2$-la ip-pa-ṭa-a*[*r*] *mi-nam a-na* 01 53
24) *lu-uš-ku-un ša* 28 15 [*i-na-d*]*i-nam* 15-**e** 30 **ib$_2$-sa$_2$**
25) 30 *a-na* 08 *ta-na-ši-ma* 04 *mi-it-har-tum iš-ti-a-at*
26) 30 *a-na* 07 *ta-na-ši-ma* 03 30 *mi-it-har-tum ša-ni-tum*

(A) I have added the areas of my two squares and 28 15 00 (is the result).
(B) (The one) square side exceeds (the other) square side by (the factor of) one seventh.
(C) You write down 08 and 07.
(D_1) You multiply 08 and 08 (and) 01 04 (is the result).
(D_2) You multiply 07 and 07.
(E) You add (the resulting) 49 and (the) 01 04 (from step D_1 and) 01 53 (is the result).
(F) The inverse of 01 53 cannot be solved.
(G) What shall I multiply by 01 53 that gives (the) 28 15 00 (from step A)? (The answer is 15 00.)
(H) The square root of (this) 15 00 is 30.
(I_1) You multiply (this) 30 by (the) 08 (written down in step C) and (the resulting) 04 00 is the first square side.
(I_2) You multiply (the same) 30 by (the) 07 (written down in step C) and (the resulting) 03 30 is the second square side.

$$x^2 + y^2 = 28\ 15\ 00 \quad , \quad y = \frac{8}{7}x$$



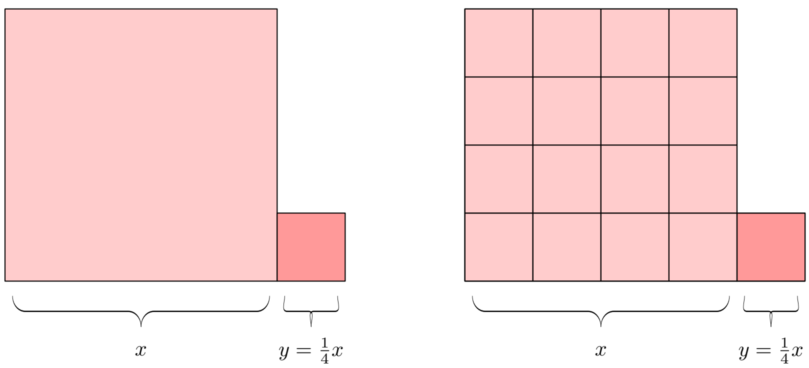
The last example is the simplest of all and needs no further explanation:

BM 13901, no. 13 (obv. ii, 36-43) (after Neugebauer (1937, 3))

36) **a-ša$_3$** *ši-ta mi-it-ha-ra-ti-ia ak-mur-ma* 28 20
37) *mi-it-har-tum ra-bi-a-at mi-it-ha-ar-tim*
38) 04 *u$_3$* 01 *ta-la-pa-at* 04 *u$_3$* 04 *tu-uš-ta-kal* 16
39) 01 *u$_3$* 01 *tu-uš-ta-kal* 01 *u$_3$* 16 *ta-ka-mar-ma* 17$^{!}$
40) **igi**-17 *u$_2$-la ip-pa-ṭa-ar mi-nam a-na* 17 *lu-uš-ku-un*
41) *ša* 28 20 *i-na-di-nam* 01 40-**e** 10 **ib$_2$-sa$_2$**
42) 10 *a-na* 04 *ta-na-ši-ma* 40 *mi-it-har-tum iš-ti-a-at*
43) 10 *a-na* 01 *ta-na-ši-ma* 10 *mi-it-har-tum ša-ni-tum*

(A) I have added the areas of my two squares and 28 20 (is the result).
(B) (The one) square side is one fourth of (the other) square side.
(C) You write down 04 and 01.
(D_1) You multiply 04 and 04 (and) 16 (is the result).
(D_2) You multiply 01 and 01.
(E) You add (the resulting) 01 and (the) 16 (from step D_1) and 17 (is the result).
(F) The inverse of 17 cannot be solved.
(G) What shall I multiply by 17 that gives (the) 28 20 (from step A)? (The answer is 01 40.)
(H) The square root of (this) 01 40 is 10.
(I_1) You multiply (this) 10 by (the) 04 (written down in step C) and (the resulting) 40 is the first square side.
(I_2) You multiply (the same) 10 by (the) 01 (written down in step C) and (the resulting) 10 is the second square side.

$$x^2 + y^2 = 28\ 20 \quad , \quad y = \frac{1}{4}x$$



1.2 Three Variables: BM 13901, no. 17

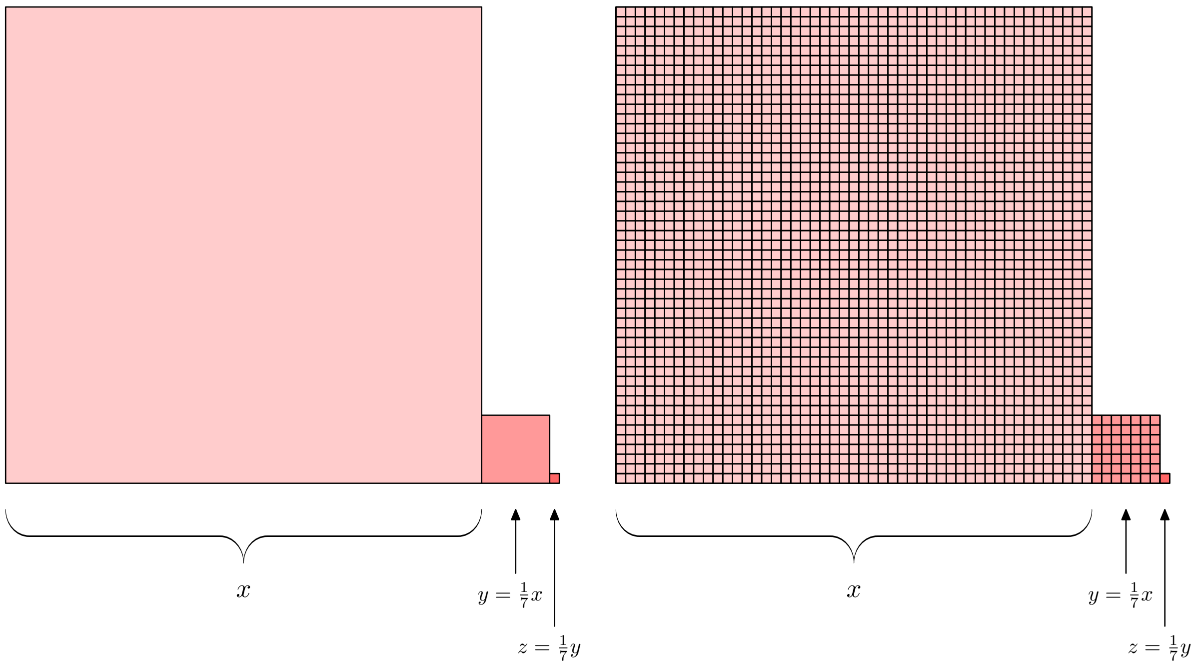
BM 13901, no. 17 (rev. i 29-38) (after Neugebauer (1937, 4))

29) [**a-ša**$_3$ *ša*]*-la-aš mi-it-ha-ra-ti-ia ak-mur-ma* 10 1[2 4]5
30) *m*[*i-i*]*t-h*[*ar-t*]*um s*[*e-b*]*i-a-at mi-it-har-t*[*im*]
31) 49 *u*$_3$ 07 *u*$_3$ 01 *ta-la-pa-at* 49 *u*$_3$ 49 [*tu-uš*]*-ta-kal* 40 01
32) 07 *u*$_3$ 07 *tu-uš-ta-kal* 49-**e** 01 *u*$_3$ 01 *tu-uš-ta-kal* 01
33) 40 01 *u*$_3$ 49 *u*$_3$ 01 *ta-ka-mar-ma* 40 [5]1 **igi**-40 51
34) *u*$_2$*-la ip-pa-ṭa-ar mi-nam a-na* 40 [5]1 *lu-uš-ku-un*
35) *ša* 10 12 45 *i-na-di-nam* 15 *ba-an-da-šu* 15-**e** 30 **ib**$_2$**-sa**$_2$
36) 30 *a-na* 49 *ta-na-ši-ma* 24 30 *mi-it-har-tum iš-ti-a-at*
37) 30 *a-na* 07 *ta-na-ši-ma* 03 30 *mi-it-har-tum ša-ni-tum*
38) 30 *a-na* 01 *ta-na-ši-ma* 30 *mi-it-har-tum ša-lu-uš-tum*

(A) I have added the areas of my three squares and 10 12 45 00 (is the result).
(B) (Each) square side is one seventh of (the preceeding) square side.
(C) You write down 49 and 07 and 01.
(D_1) You multiply 49 and 49 (and) 40 01 (is the result).
(D_2) You multiply 07 and 07 (and) 49 (is the result).
(D_3) You multiply 01 and 01 (and) 01 (is the result).
(E) You add (the) 40 01 and 49 and 01 (from steps D_1 to D_3) and 40 51 (is the result).
(F) The inverse of 40 51 cannot be solved.
(G) What shall I multiply by 40 51 that gives (the) 10 12 45 00 (from step A)? Its quotient (i.e. the answer) is 15 00.
(H) The square root of (this) 15 00 is 30.
(I_1) You multiply (this) 30 by (the) 49 (written down in step C) and (the resulting) 24 30 is the first square side.
(I_2) You multiply (the same) 30 by (the) 07 (written down in step C) and (the resulting) 03 30 is the second square side.
(I_3) You multiply (the same) 30 by (the) 01 (written down in step C) and (the resulting) 30 is the third square side.

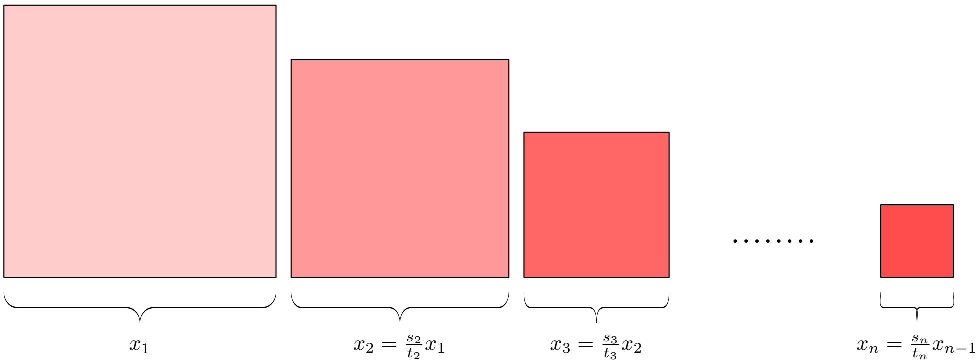
$$x^2 + y^2 + z^2 = 10\ 12\ 45\ 00 \quad , \quad y = \frac{1}{7}x\,,\ z = \frac{1}{7}y$$

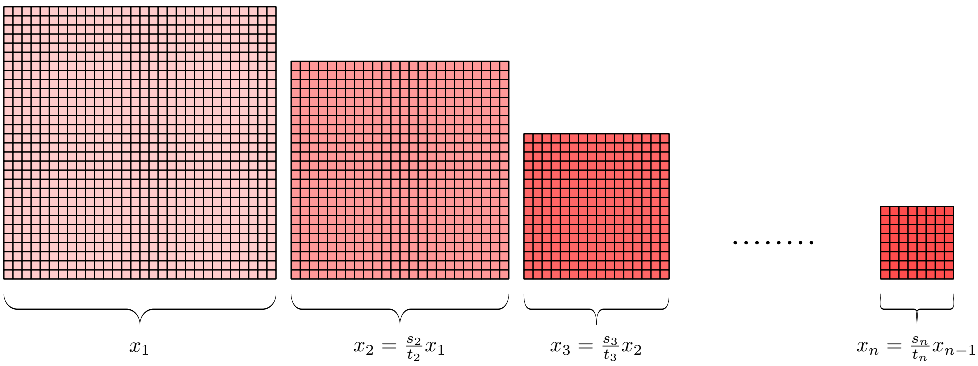
Again one can think of the unknown square sides (this time three) as subdivided into equal little pieces. Since the first square side is $07 \cdot 07 = 49$ times as long as the third one and the second square side is 07 times as long as the third one, the third unknown square is chosen as "little square". Its side length fits 49 times into the first side length, 07 times into the second side length, and of course 01 times into the third one.



Correspondingly, in step (C) the numbers 49, 07, and 01 are recorded. In the steps (D_1) to (D_3) these numbers are squared and their squares added in step (E) in order to obtain the total number of little squares whose combined area is again the given total area, here 10 12 45 00. Division of this total area by the total number of little squares (steps F and G, cf Remark 1.1) gives the area of one little square and by extracting its square root (step H) one gets its side length which is again 30. Multiplying this length by the respective numbers of occurrence one obtains the three formerly unknown square sides (steps I_1 to I_3).

1.3 The General Case





The general set of equations is

$$\sum_{i=1}^{n} x_i^2 = A \tag{1}$$

$$x_i = \frac{s_i}{t_i} x_{i-1} \qquad \text{for} \quad 2 \leq i \leq n \tag{2}$$

where $s_i, t_i \in \mathbb{N} \setminus \{0\}$, and $A > 0$ are given constants and the x_i are the unknowns asked for. (Without loss of generality, for every i, s_i and t_i can be assumed to have no common divisors, i.e. the fraction $\frac{s_i}{t_i}$ cannot be simplified.) This leads to

$$x_i = \left(\prod_{j=2}^{i} \frac{s_j}{t_j} \right) x_1 \qquad \text{for} \quad 2 \leq i \leq n \tag{3}$$

The solution procedure is then as follows.

Each of the unknown squares (side lengths x_i) is subdivided into little squares of side length $u = \left(\prod_{i=2}^{n} \frac{1}{t_i} \right) x_1$ and therefore of area $S = \left[\left(\prod_{i=2}^{n} \frac{1}{t_i} \right) x_1 \right]^2$. This length u fits into x_i, the side length of the i-th square, p_i times, where

$$p_1 = \prod_{j=2}^{n} t_j \tag{4}$$

$$p_i = \frac{s_i}{t_i} p_{i-1} \qquad \text{for} \quad 2 \leq i \leq n \tag{5}$$

and therefore

$$p_i = \left(\prod_{j=2}^{i} s_j \right) \left(\prod_{k=i+1}^{n} t_k \right) \qquad \text{for} \quad 1 \leq i \leq n. \tag{6}$$

These are the numbers that would be written down in the general step (C).

This means that the i-th square (side length x_i) is built up from p_i^2 little squares.

These are the numbers that would be computed in the general steps (D_1) to (D_n).

The total amount Q of little squares is therefore

$$Q = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} \left[\left(\prod_{j=2}^{i} s_j \right) \left(\prod_{k=i+1}^{n} t_k \right) \right]^2 . \tag{7}$$

This number would be computed in the general step (E).

The total area of these Q little squares is nothing else but the total area of the n unknown squares, i.e. A. The area of each little square is therefore $\frac{A}{Q}$

(performed by the general steps (F) and (G), unless of course Q happens to be sexagesimally regular in which case the division could be performed directly)

and its side length is $\sqrt{\frac{A}{Q}}$.

(general step H)

The i-th square side x_i is made up from p_i copies of this length. So we finally get

$$x_i \;=\; p_i \sqrt{\frac{A}{Q}} \qquad \text{for} \quad 1 \leq i \leq n. \tag{8}$$

(general steps I_1 to I_n)

Remark 1.2 If the $p_i (1 \leq i \leq n)$ have a common divisor, say d, the number of little squares can be reduced. For it is then possible to subdivide each x_i into only $\frac{p_i}{d}$ equal pieces of length du. This leads to a total of only $\frac{Q}{d^2}$ little squares each of which has side length du.

Remark 1.3 If, on the other hand, one (artificially) replaced all the p_i with qp_i where q is a positive integer > 1, i.e. multiplied the number of little squares by q^2, one would just scale the problem up. It would complicate the solution process since one would have to deal with larger numbers, but the result would naturally be the same.

Remark 1.4 The method generalizes in an obvious way to the case of subtracting (and/or adding) the squares, i.e. to the case where the quadratic part of the system looks like $x^2 - y^2 = A$ or $x^2 + y^2 - z^2 = A$ etc. It also generalizes in an obvious way to the case where integer multiples of the squares occur, e.g. equations like $3x^2 + 4y^2 - 2z^2 = A$ etc. All this is covered in general by type IIId (section 6).

1.4 Four Variables: BM 13901, no. 15

BM 13901, no. 15 (rev. i 12-22) differs from the general setting described in section 1.3 above and from the given examples in two and three variables in that it is somewhat simpler put. Namely, the third and fourth unknown square side is not given as a fractional multiple of the second and third one, respectively, but, like the second one, as a fraction of the first one:

$$x^2 + y^2 + z^2 + w^2 = 27\ 05 \quad , \quad y = \frac{2}{3}x \, , \, z = \frac{1}{2}x \, , \, w = \frac{1}{3}x$$

This saves one computational step, but otherwise the procedure is the same. Observe that in the solution procedure the square sides are subdivided into 01 00 and 40 and 30 and 20 equal pieces, respectively, whereas 6, 4, 3, and 2 would have done the trick. Cf Remark 1.3 above. The reason is probably that the cuneiforms 01, 40, 30, and 20 are also the sexagesimal representations of 1, $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{3}$, respectively.

Remark 1.5 It is possible (and maybe even likely), however, that 01 00 and 40 and 30 and 20 are to be interpreted as 01 and 00▲40 and 00▲30 and 00▲20 instead, and that the problem was dealt with by saying that $x^2 + y^2 + z^2 + w^2 = \left(1^2 + (\frac{2}{3})^2 + (\frac{1}{2})^2 + (\frac{1}{3})^2\right) x^2 = 27\ 05$ and solving this for x directly instead of by the method above. Of course this would not be possible if the fractions involved were such that the analog term (i.e. the one replacing $\left(1^2 + (\frac{2}{3})^2 + (\frac{1}{2})^2 + (\frac{1}{3})^2\right)$) happened to be sexagesimally non-regular, as it was the case in the examples above (involving $\frac{6}{7}$ and $\frac{8}{7}$ and $\frac{1}{7}$). So one might be tempted to argue that likely the general procedure described in section 1.3 was reserved to such cases.

The text is severely damaged and almost all the parts relevant for its interpretation have been restored by O. Neugebauer. In the transliteration below, the restored part is printed in grey shade.

BM 13901, no. 15 (rev. i 12-22) (after Neugebauer (1937, 3-4))

12) [**a-ša$_3$** *er-be*$_2$*-e mi-it-ha-r*]*a-ti-ia ak-mur-ma* 27 05
13) [*mi-it-har-tum ši-ni-pa-at mi-ši*]*-il*$_5$ *ša-lu-uš-ti mi-it-har-tim*
14) [01 *u*$_3$ 40 *u*$_3$ 30 *u*$_3$ 20 *t*]*a-la-pa-at* 01 *u*$_3$ 01 *tu-uš-ta-kal* 01
15) [40 *u*$_3$ 40 *tu-uš-ta-kal* 26] 40-**e** 30 *u*$_3$ 30 *tu-uš-ta-kal* 15
16) [20 *u*$_3$ 20 *tu-uš-ta-kal* 06] 40 *u*$_3$ 15 *u*$_3$ 26 40 *u*$_3$ 01
17) [*ta-ka-mar* **igi** 01 48] 20-**e** *u*$_2$*-la ip-pa-ṭa-ar*
18) [*mi-nam a-na* 01 48 20] *lu-uš-ku-un ša* 27 05 *i-na-di-nam*
19) [15-**e** 30 **ib$_2$-sa$_2$** 30 *a-na*] 01 *ta-na-ši-ma* 30 *mi-it-har-tum iš-ti-a-at*
20) [30 *a-na* 40 *ta-na-ši-m*]*a* 20 *mi-it-har-tum ša-ni-tum*
21) [30 *a-na* 30 *ta-na-ši-m*]*a* 15 *mi-it-har-tum ša-lu-uš-tum*
22) [30 *a-na* 20 *ta-na-ši-ma* 1]0 *mi-it-har-tum re-bu-tum*

(A) I have added the areas of my four squares and 27 05 (is the result).
(B) A square side is two thirds, (respectively) half, (respectively) one third, of (the first) square side.
(C) You write down 01 00 and 40 and 30 and 20.
(D$_1$) You multiply 01 00 and 01 00 (and) 01 00 00 (is the result).
(D$_2$) You multiply 40 and 40 (and) 26 40 (is the result).
(D$_3$) You multiply 30 and 30 (and) 15 00 (is the result).
(D$_4$) You multiply 20 and 20 (and 06 40 is the result).
(E) You add (the) 06 40 and 15 00 and 26 40 and 01 00 00 (from steps D$_1$ to D$_4$ and 01 48 20 is the result).
(F) The inverse of 01 48 20 cannot be solved.
(G) What shall I multiply by 01 48 20 that gives (the) 27 05 (from step A)? (The answer is 00 ▲ 15.)
(H) The square root of (this) 00 ▲ 15 is 00 ▲ 30.
(I$_1$) You multiply (this) 00 ▲ 30 by (the) 01 00 (written down in step C) and (the resulting) 30 is the first square side.
(I$_2$) You multiply (the same) 00 ▲ 30 by (the) 40 (written down in step C) and (the resulting) 20 is the second square side.
(I$_3$) You multiply (the same) 00 ▲ 30 by (the) 30 (written down in step C) and (the resulting) 15 is the third square side.
(I$_4$) You multiply (the same) 00 ▲ 30 by (the) 20 (written down in step C) and (the resulting) 10 is the fourth square side.

2 The Type Ib

This time, along with the sum of the unknown squares' areas, an increment is given by which the subsequent side lengths differ.[3] In the following example with three unknowns ($n = 3$) this reads as

$$x^2 + y^2 + z^2 = A \quad , \quad y = x + b \, , \, z = y + b$$

where x, y and z are the unknown side lengths, A is the given combined area of the three squares and b, the increment, is a (for the time being) positive number. (Actually, it is a length, but represented by a number without units of measure.)

2.1 Three Variables

BM 13901, no. 18 (rev. i, 39-49)[4] (after Neugebauer (1937, 4))

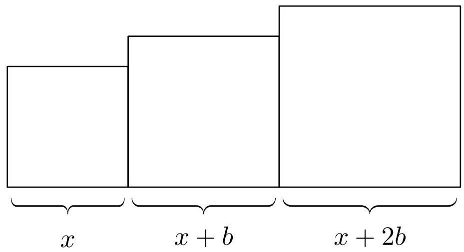
39) **a-ša$_3$** *ša-la-aš mi-it-ha-ra-ti-ia ak-mur-ma* 23 20
40) *mi-it-har-tum* **ugu** *mi-it-har-tim* 10 *i-te-er*
41) 10 *ša i-te-ru a-na* 01 *ta-na-ši* 10 *a-na* 02 *ta-na-ši* 20 **u$_3$** 20
42) 06 40-**e** 10 **u$_3$** 10 *tu-uš-ta-kal* 01 40 *a-na* 06 40 *tu-ṣa-ab* 08 20
43) *lib$_3$-ba* 23 20 *ta-na-sa$_3$-ah-ma* 15 *a-na* 03 *mi-it-ha-ra-*[*ti*]
44) *ta-na-ši* 45 *ta-la-pa-at* 10 **u$_3$** 20 *ta-ka-mar-ma*
45) 30 **u$_3$** 30 *tu-uš-ta-kal* 15 *a-na* 45 *tu-ṣa-ab-ma*
46) 01-**e** 01 **ib$_2$-sa$_2$** 30 *ša tu-uš-ta-ki-lu ta-na-sa$_3$-ah-ma* 30 *ta-la-pa-at*
47) **igi**-03 *mi-it-ha-ra-ti* 20 *a-na* 30 *ta-na-ši* 10 *mi-it-har-tum*
48) 10 *a-na* 10 *tu-ṣa-ab-ma* 20 *mi-it-har-tum* **ki**-2 10 *a-na* 20
49) *tu-ṣa-ab-ma* 30 *mi-it-har-tum* **ki**-3

(A) I have added the areas of my three squares and 23 20 (was the result).
(B) (Each) square side exceeds (the preceeding) square side by 10.
(C) The 10 by which (the square sides) exceed (one another) you take once.
(D) You take (this) 10 twice (as well).
(E) (The resulting) 20 and (the resulting) 20 (multiplied result) in 06 40.
(F) You multiply the 10 (from step C) and the 10 (from step C).
(G) You add (the resulting) 01 40 to the 06 40 (from step E).
(H) You subtract (the resulting) 08 20 from 23 20 (from line A) and
(I) you multiply (the resulting) 15 00 by 03 (which is the number of the) squares.
(J) You record (the resulting) 45 00.
(K) You add 10 (obtained in step C) and 20 (obtained in step D) and
(L) you multiply (the resulting) 30 and (the resulting) 30.
(M) You add (the resulting) 15 00 to the 45 00 (recorded in line J) and
(N) the square root of (the resulting) 01 00 00 is 01 00.
(O) (From this 01 00) you subtract the 30 you have squared (in step L) and

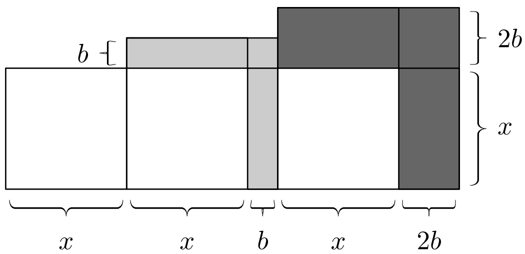
[3] The material for sections 2.1 and 2.2 is taken from Brunke (2017, 6-9, 12-14).
[4] See Høyrup (2002, 108-110).

(P) you record (the resulting) 30.
(Q) The inverse of 03 (which is the number of the) squares, (namely) 00 ▲ 20, you multiply with (the) 30 (recorded in line P).
(R) (The resulting) 10 is the (first) square side.
(S) You add (the) 10 (given in line B) to (this) 10 (obtained in step R) and (the resulting) 20 is the second square side.
(T) You add (the) 10 (given in line B) to (this) 20 (obtained in step S) and (the resulting) 30 is the third square side.

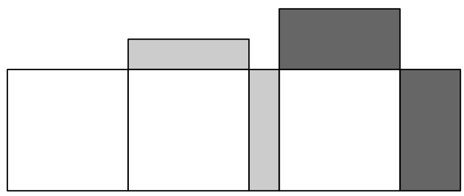
The total area A, i.e. the sum of the areas of the three squares (here $A = 23\ 20$ as stated in line A), can be depicted as follows.



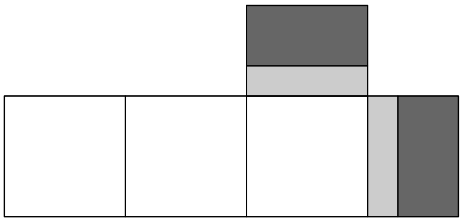
The surplusses of the second and the third square over the first one are given different shades of grey.



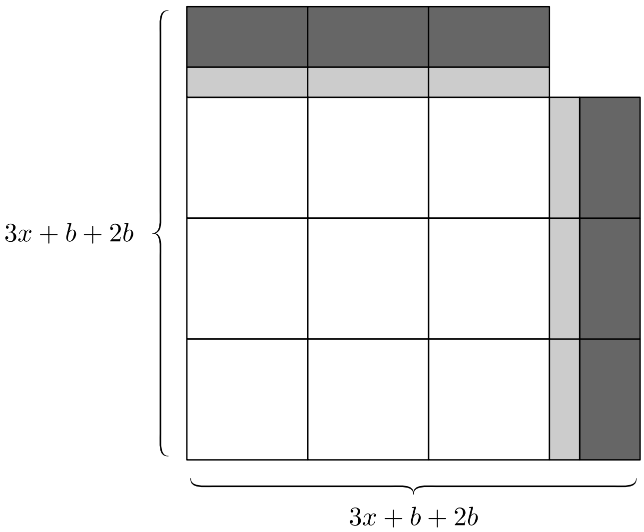
The amount b by which the subsequent square sides increase (here $b = 10$ as stated in line B) is multiplied by 01 (step C) as well as by 02 (step D) resulting in b and $2b$, respectively. These are the amounts by which y and z, respectively, exceed x. Both of them are squared (steps E and F) which gives the areas $(2b)^2$ and b^2 of the little grey-shaded squares in the upper right corners of the third and the second square, respectively. These two areas are taken together (step G) and removed from the figure (step H):



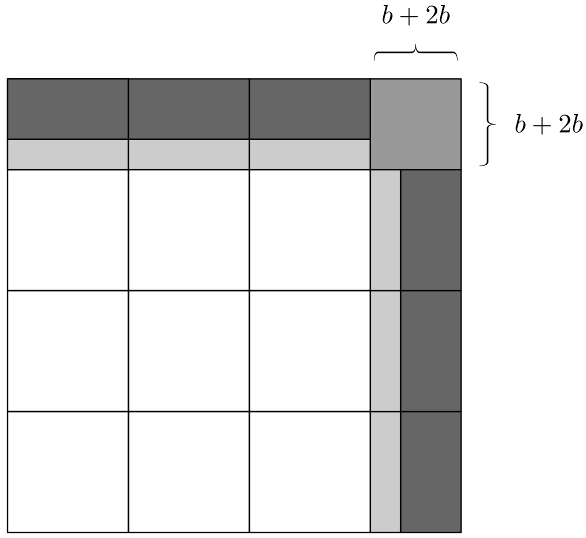
The area of the resulting figure is $A' := A - [b^2 + (2b)^2]$. This figure can be rearranged in the following way without changing the total area:



Be aware that this rearrangement is not mentioned or even indicated in the text (as such rearrangement operations never are, as it seems). In the next step (I) the whole arrangement is taken three (which is the number of unknown squares) times. The result may be organized as shown in the following picture.



The area of the new figure is $A'' := 3A'$ (here with value 45 00 as stated in line J). The figure itself is a square with side length $3x + b + 2b$, with a square of side length $b + 2b$ missing in the upper right corner. The value for $b + 2b$ is computed in step (K) and squared in step (L) which gives the area $(b + 2b)^2$ (here 15 00) of the square missing in the upper right corner. Adding the square with side length $b + 2b$ to the figure (step M) therefore results in a complete square with area $A''' := A'' + (b + 2b)^2$ (here 01 00 00) and side length $3x + b + 2b$

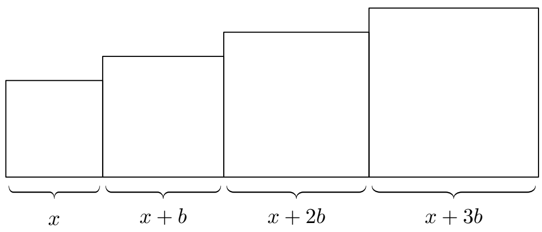


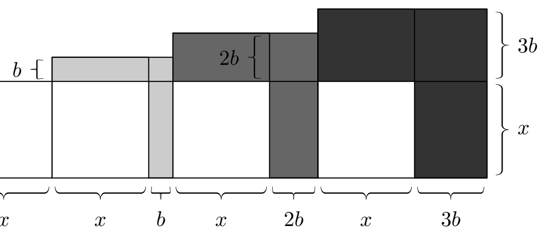
whence extracting the square root from A''' (step N) gives $3x+b+2b$. By subtraction of $b+2b$ (step O) we get $3x$ and the third part of this (step Q) is x, the side of the first (smallest) of the three squares. Successive addition of b, the surplus of y over x and of z over y, (steps S and T) renders the sides of the second and third square, respectively.

Remark 2.1 Observe that the number 03 by which two of the intermediate results are multiplied (step I) and divided (step Q), is explicitely referred to as "the (number of the) squares" (lines 43 and 47: 03 *mitharāti*). This shows the awareness of the fact that the analog problems (with four, five, six ... squares) can be solved by means of the very same method, and therefore of the general validity of the solution algorithm. This insight goes back to Neugebauer (1937, 13) who claims that "aus der ganzen Formulierung deutlich hervorgeht, daß hier der Fall von drei Unbekannten nur als Spezialfall einer analogen Aufgabe ... für eine beliebige Anzahl n von Unbekannten angesehen wurde." For the case $n = 4$ see section 2.2 below, and for $n = 2$ see appendix A. The most general case is discussed in section 2.4 below.

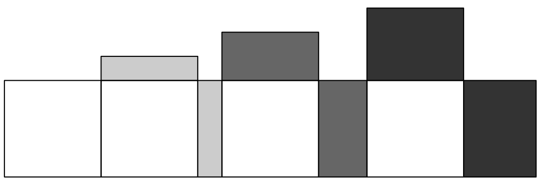
2.2 Hypothetical Example: The Case of Four Variables

The following sequence of pictures shows how the method for Type Ib works in the case of four squares. The setting of the four squares is the following:

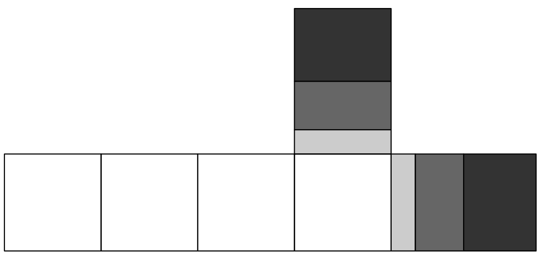




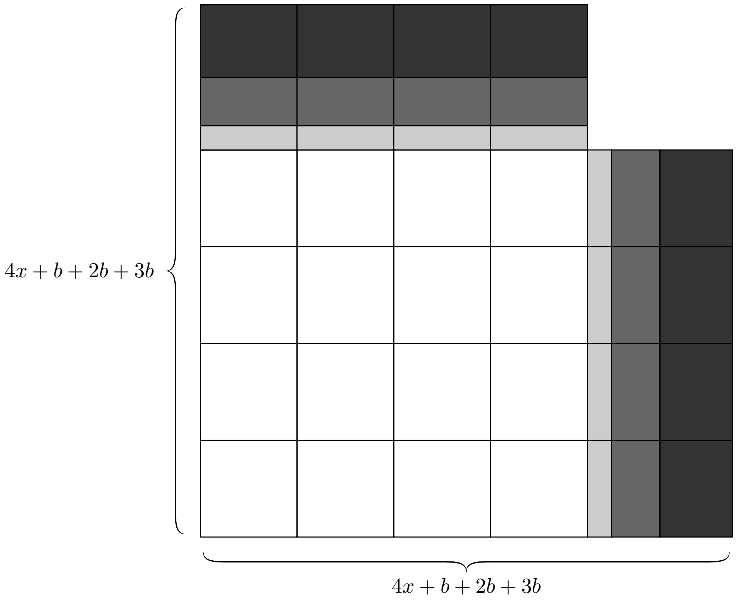
Now the three little squares with the sides b, $2b$, and $3b$, respectively, are removed from the top right corners of the second, third, and fourth square



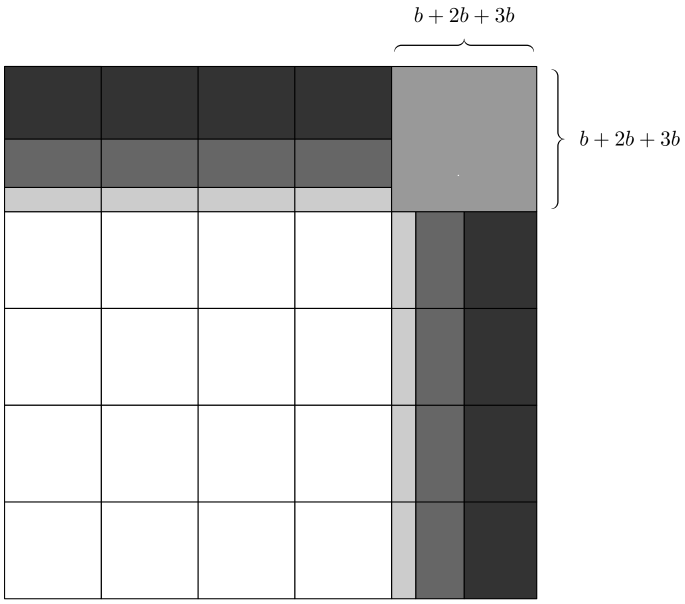
and the remaining parts are rearranged:



Next, the whole arrangement is taken four (which now is the number of squares) times and the result is organized as follows:



Then a square with side length $b + 2b + 3b$ is added in order to obtain a complete square with side length $4x + b + 2b + 3b$.



Then one would extract the square root from the complete square's area in order to obtain $4x + b + 2b + 3b$. The remaining steps are obvious.

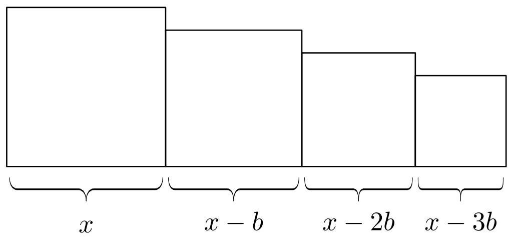
Remark 2.2 It is straight forward how the method generalizes to the case of various increments (i.e. $y = x + b$, $z = y + c$, $w = z + d$). In the procedure (for the general case of which see section 2.4) as well as in the diagrams the replacements $2b \to b + c$ and $3b \to b + c + d$ will do the trick. It is also obvious how the generalisation to arbitrary n works; see section 2.4 for details.

2.3 The Case of Negative Increments

Here we consider the case where the subsequent square sides *decrease* in size which can be described in terms of "negative increments" in modern terminology. This is exemplified by the case $n = 4$ as in section 2.2. It is described by the set of equations ($b > 0$)

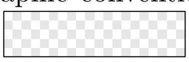
$$x^2 + y^2 + z^2 + w^2 = A \quad , \quad y = x - b\,,\, z = y - b\,,\, w = z - b$$

and depicted by the figure

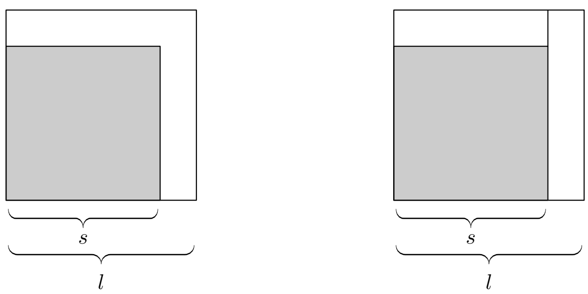


wich has total area A.

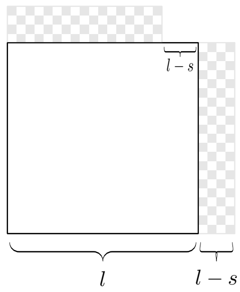
Remark 2.3 Again, the method generalizes in an obvious way to the case of various increments (i.e. $y = x - b$, $z = y - c$, $w = z - d$), cf remark 2.2, and to the case of arbitrary n for which see in detail section 2.4.

In order to perform a graphic cut-and-paste solution analogous to the one in sections 2.1 and 2.2 above, and also in order to cope for the fact that there were no "negative" magnitudes in Mesopotamian mathematics but rather (positive) "magnitudes-to-be-subtracted" we establish a graphic convention to deal with such magnitudes. Subsequently, a "pixelated" rectangle like  denotes a rectangular area-to-be-subtracted. Whenever such a pixelated rectangle occurs in a drawing it means that its area is to be subtracted from the figure composed of "solid" rectangles and squares.

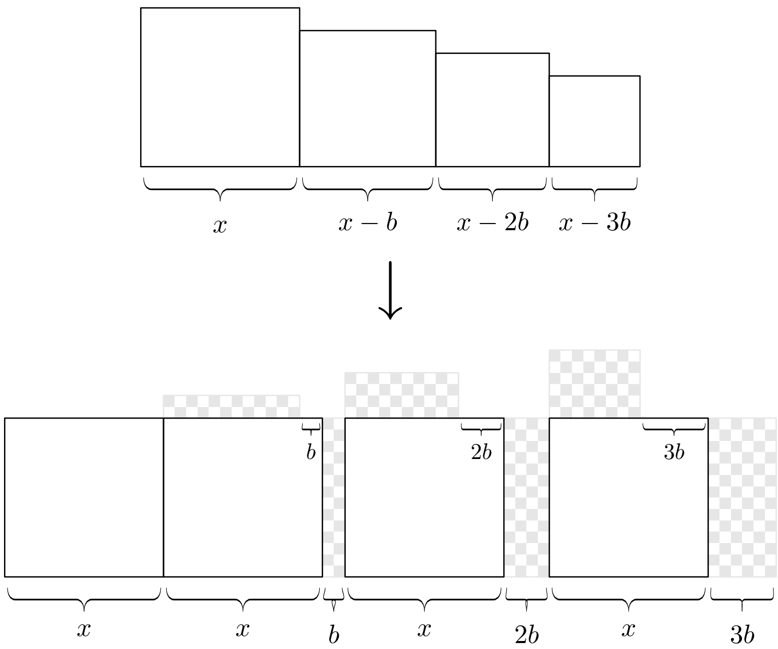
For example, a square with side length s can be obtained from a larger square with side length $l > s$ by subtracting a rectangle with side lengths s and $l - s$ (the horizontal one in the right-hand drawing below) and a rectangle with side lengths l and $l - s$ (the vertical one in the right-hand drawing below):



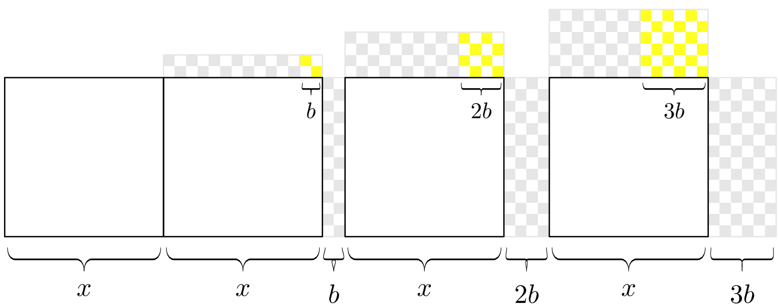
This will be depicted by adding pixelated (i.e. "negative") rectangles of the respective sizes to the bigger square (with side length l):



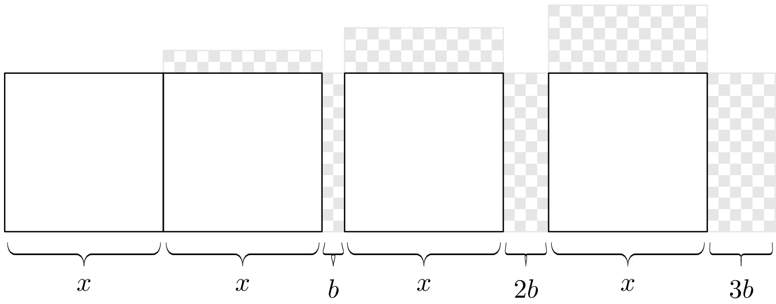
In particular, a square with side length $x-b$ is obtained from the square with side length x by adding a pixelated rextangle with side lengths $x-b$ and b (horizontal) and one with side lengths x and b (vertical). Analogously for squares with side length $x-2b$ or $x-3b$. Therefore, using pixelated rectangles, the figure depicting the problem at hand (with total area A) looks like this:



Now, subtracting three squares with the side lengthss b, $2b$, and $3b$ from the upper right corners of the second, third, and fourth square, respectively, (as in section 2.2) amounts to *adding* the corresponding pixelated squares (yellow):

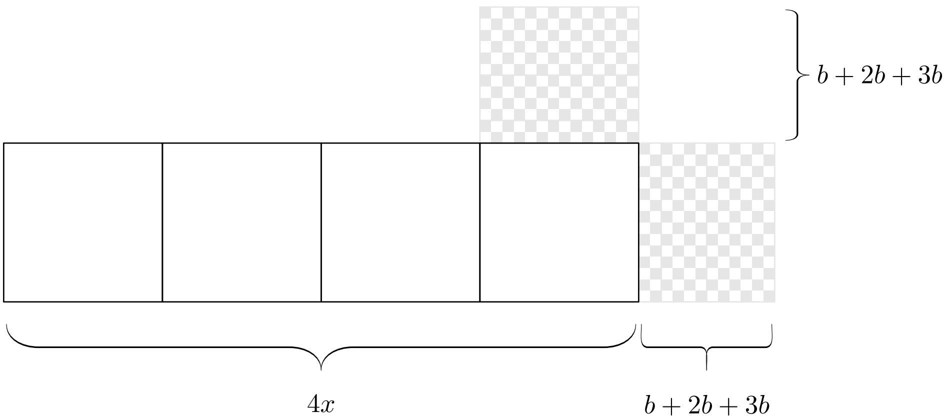


The area of the new figure

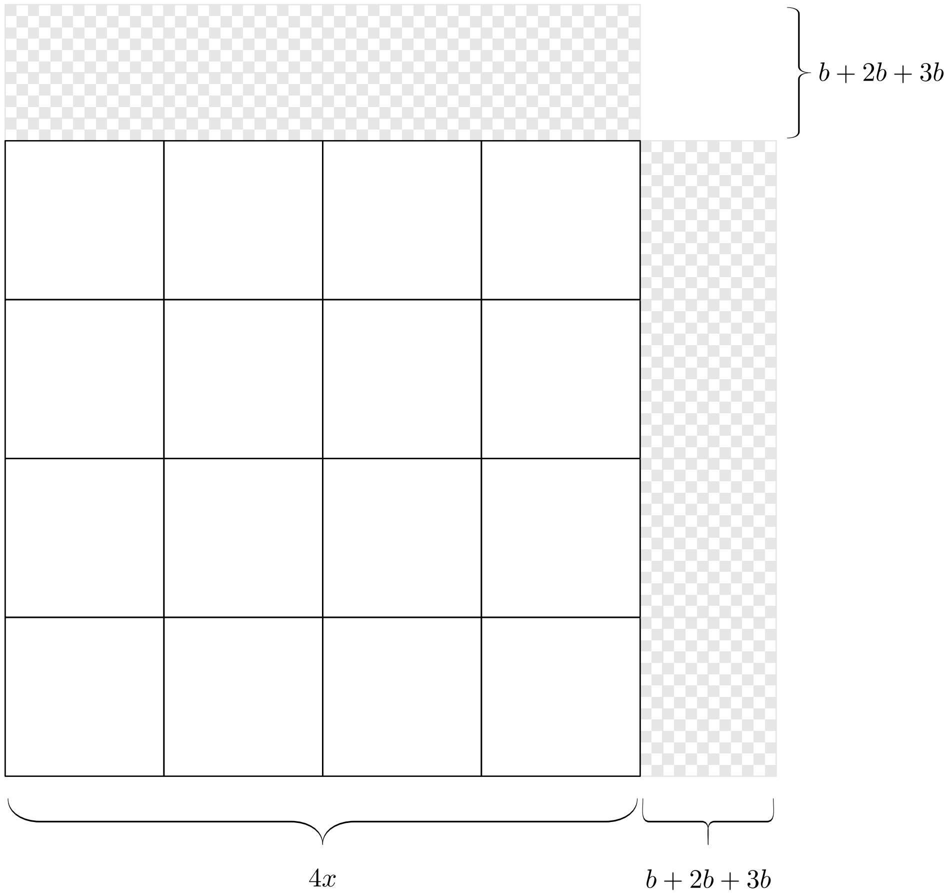


is $A' := A - (b^2 + (2b)^2 + (3b)^2)$.

Rearranging the figure to

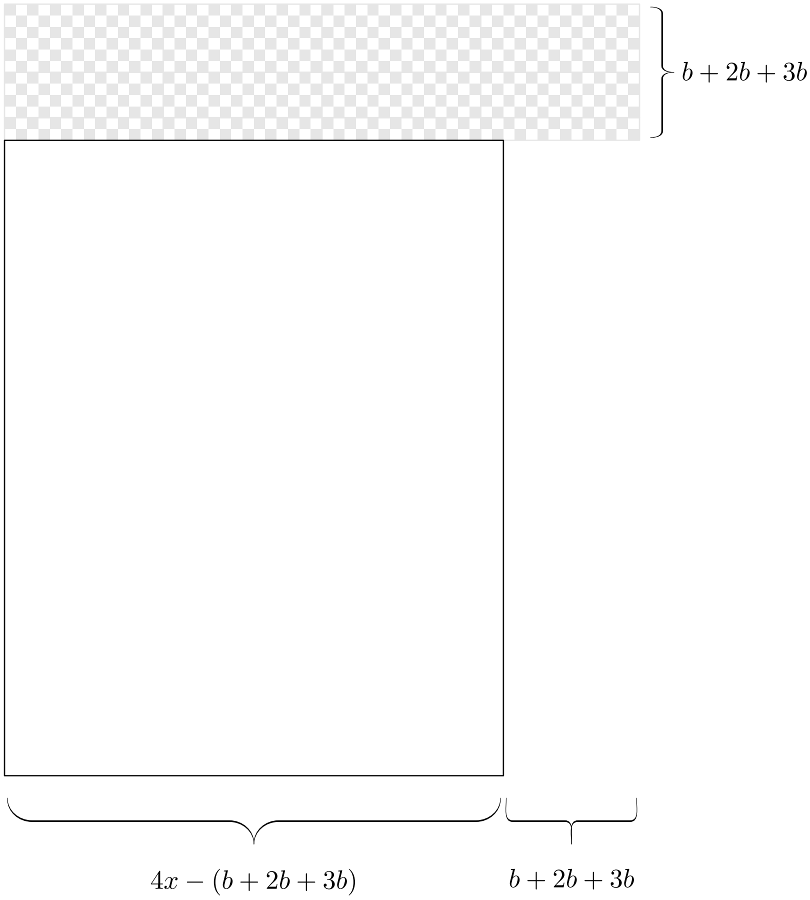


and taking it $n = 4$ (i.e. the number of unknown squares) times:

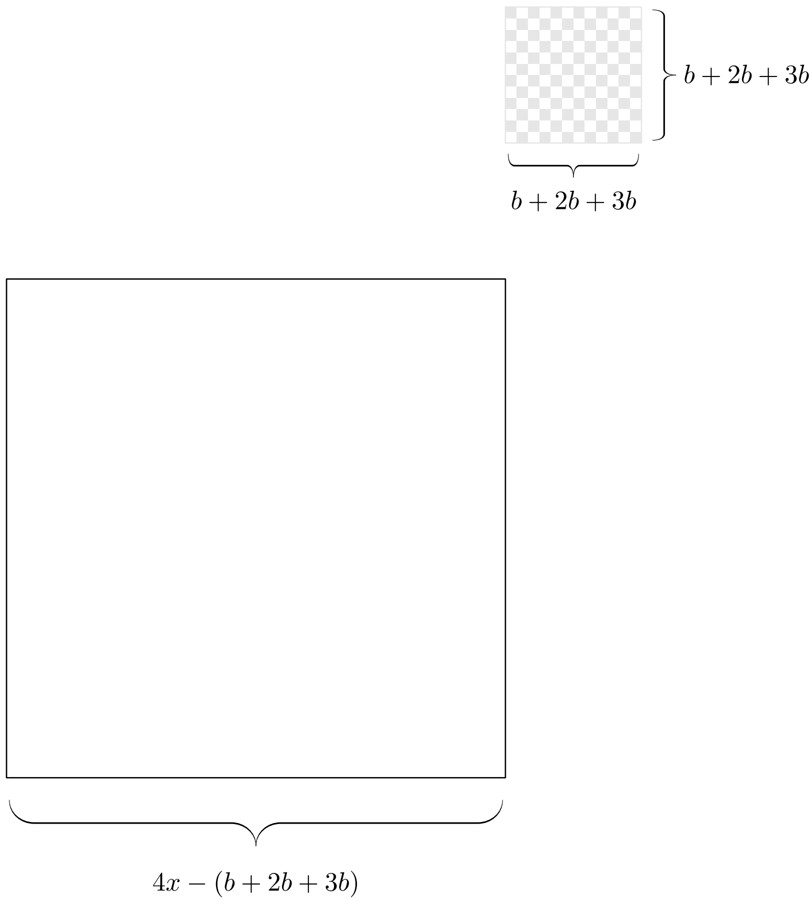


results in a “solid” square with side length $4x$ (nx in the general case), plus two “negative rectangles”. The total area of this figure is $A'' := 4A'$ (nA' in the general case).

Now the remaining pixelated rectangles are finally dealt with, i.e. subtracted from the “solid” square. Subtracting first the vertical one leaves

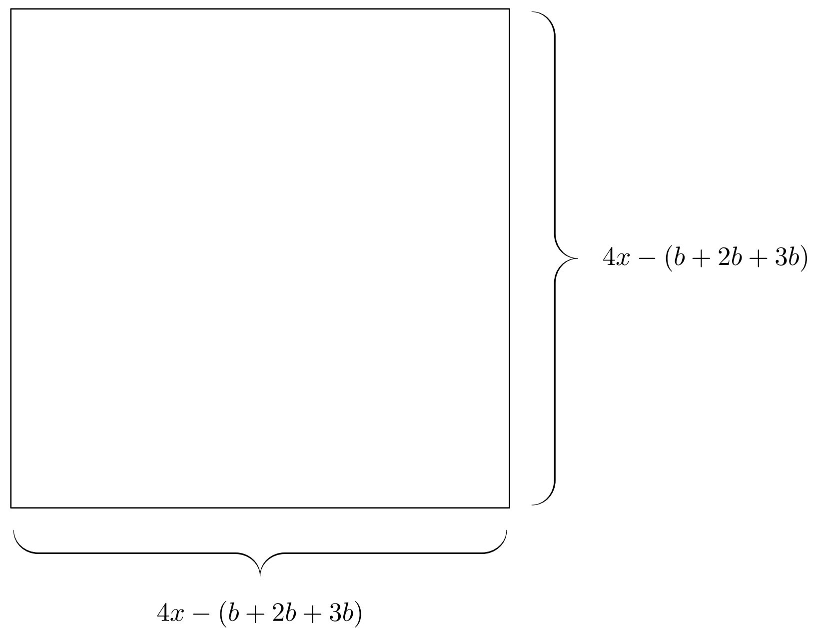


and then removing the horizontal one we end up with



which is a solid square with side length $4x - (b + 2b + 3b)$ plus a “negative” square with side length $b + 2b + 3b$. The total area of the figure is still A''. Adding a square with side length $b+2b+3b$ (as in section 2.2) of course results in a figure with area $A''' := A'' + (b+2b+3b)^2$.

This adding process eliminates the corresponding negative square and thus results in a solid square with side length $4x - (b + 2b + 3b)$ and therefore area $(4x - (b + 2b + 3b))^2$:



From this it follows that $A''' = (4x - (b + 2b + 3b))^2$ from which x (and subsequently y, z and w) can be computed, exactly as in section 2.1: extracting the square root from A''' gives $4x - (b + 2b + 3b)$, etc.

Remark 2.4 The method generalizes in an obvious way to the case where positive and negative increments occur at the same time. In the formal description (section 2.4 below) this fact is covered by allowing the b_i to take negative as well as positive values.

Remark 2.5 Of course, the problem could as well be solved by reversing the order of the squares and using positive increments. However, the solution procedure of type Ic example Strssbg. 363, obv. 1-12 (see section 4.2.2 below) indicates that this was not the way it was done. Also, in the concluding statements of the procedures (when the final results are presented) the squares are always referred to as "first, second, third, ... square" which shows that the order was considered fix and not supposed to be reversed in the process. (For this ordering according to *decreasing* side lengths see in particular the Type Ib^{s} examples YBC 4714, problems 2 and 3, in section 3.)

2.4 The General Case

The general set of equations is

$$\sum_{i=1}^{n} x_i^2 = A \tag{9}$$

$$x_i = x_{i-1} + b_i \qquad \text{for} \quad 2 \leq i \leq n \tag{10}$$

where $b_i \in \mathbb{R}$ and $A > 0$ are given constants and the x_i are the unknowns asked for. This leads to

$$x_i = x_1 + \Delta_i \qquad (1 \leq i \leq n) \tag{11}$$

where

$$\Delta_1 = 0 \qquad \text{and} \qquad \Delta_i = \sum_{j=2}^{i} b_j \quad \text{for} \quad 2 \leq i \leq n\,. \tag{12}$$

Thus the i-th unknown square (with side length x_i) is made up from a "base square" of side length x_1 and (in the case $i \geq 2$) two additional rectangles (one horizontal, one vertical) with side lengths x_1 and Δ_i, and an additional square of side length Δ_i. The total area of the arrangement of all these squares is just A.

The first step towards the solution is to remove from this arrangement all the objects that are completely determined by the given magnitudes (i.e., the b_i), namely the additional squares with side lengths Δ_i for all $2 \leq i \leq n$. The remaining arrangement has the area

$$A' \;=\; A - \sum_{i=2}^{n} \Delta_i^2 \;=\; A - \sum_{i=2}^{n} \left(\sum_{j=2}^{i} b_j \right)^2 . \tag{13}$$

Now, rearrange all the n base squares (side lengths x_1) into a horizontal row. Arrange all the horizontal additional rectangles into one long rectangular stripe with widths x_1 and length $L \;=\; \sum_{i=2}^{n} \Delta_i$. Do the same with all the vertical additional rectangles.

So far, the area of the new arrangement is still A'. Take this arrangement n (the number of unknown squares) times. The resulting n^2 squares of side length x_1 are arranged into a big square with side length nx_1. The resulting n horizontal and n vertical rectangular stripes are added on the right and on top of the new big square, respectively, each stripe fitting exactly next to (respectively above) one of the side and top sub-squares (side length x_1) of the big square.

The resulting object has the shape of a square of side length $nx_1 + L$ with a square of side length L missing in the top right corner, and its area is

$$A'' \;=\; nA'\,. \tag{14}$$

Adding the mentioned missing square (which of course has the area L^2) one obtains the complete square of side length $nx_1 + L$. Its area is $A''' \;:=\; A'' + L^2$. Therefore we have $\sqrt{A'''} = nx_1 + L$ and from this

$$x_1 = \frac{\sqrt{A'''} - L}{n}\,. \tag{15}$$

The remaining solutions then are $x_2 = x_1 + b_2$, $x_3 = x_1 + b_2 + b_3$, ... Or in compact notation:

$$x_i \;=\; x_1 + \sum_{j=2}^{i} b_j \qquad \text{for } 2 \leq i \leq n\,. \tag{16}$$

Remark 2.6 For details concerning the case that some or all of the b_i are negative see Remark 4.4. This applies here since Ib can be derived as a special case of Ic in an obvious way.

3 The Type Ib^{s}

This type of problem is similar to the type Ib but now not only the sides of the squares are unknown, but also the increment by which successive square sides exceed one another. The latter is, however, neither asked for in the statement of the problem, nor is its value given in the solution in the end. One may thus assume that it was considered an auxiliary variable only. Examples are YBC 4714, problems 2 and 3 (Neugebauer, 1935, 487, 492). To my knowledge there is no attestation of the actual solution procedure known so far, but it seems plausible that it might have been conducted along the lines described below.

YBC 4714, no. 2 (obv. i 5-13) (after Neugebauer (1935, 487))

5)	**a-ša$_3$ ib$_2$-sa$_2$** 04-**e**	The areas of 4 squares
6)	**g̃ar-g̃ar**-*ma* 01 30	added and 01 30 00 (is the result)
7)	**ib$_2$-sa$_2$** 04-**e**	The 4 square sides
8)	**g̃ar-g̃ar**-*ma* 02 20	added and 02 20 (is the result)
9)	**i**[**b$_2$-s**]**a$_2$** 04-**e en-nam**	The 4 square sides (are) what?
10)	50 [**nindan**] 01-**e**	50 [nindan] the first.
11)	40 **nindan** 02-**e**	40 nindan the second.
12)	30 **nindan** 03-**e**	30 nindan the third.
13)	20 **nindan** 04-**e**	20 nindan the fourth.

YBC 4714, no. 3 (obv. i 14-24) (after Neugebauer (1935, 487))

14)	**a-ša$_3$ ib$_2$-sa$_2$** 06-**e**	The areas of 6 squares
15)	**g̃ar-g̃ar**-*ma* 01 52 55	added and 01 52 55 (is the result)
16)	**ib$_2$-sa$_2$** 06-**e**	The 6 square sides
17)	**g̃ar-g̃ar**-*ma* 03 15	added and 03 15 (is the result)
18)	**ib$_2$-sa$_2$** 06-**e en-nam**	The 6 square sides (are) what?
19)	45 **nindan** 01-**e**	45 nindan the first.
20)	40 **nindan** 02-**e**	40 nindan the second.
21)	35 **nindan** 03-**e**	35 nindan the third.
22)	30 **nindan** 04-**e**	30 nindan the fourth.
23)	25 **nindan** 05-**e**	25 nindan the fifth.
24)	20 **nindan** 06-**e**	20 nindan the sixth.

In view of the order in which the final results are given in both examples (lines 10-13, and 19-24, respectively) we seem to be dealing with the case of *negative increments* here (cf remark 2.5 above). In the following general treatment as well the case for positive as for negative increment will be demonstrated, starting with the further.

3.1 The case of positive increment

While the explanation covers the general case for arbitrary n, the drawings depict the special situation for $n = 4$ (like in YBC 4714, no. 2). It corresponds to the situation given in section 2.2, the only difference being that now the (constant) difference between the side lengths of subsequent squares is unknown. The following abbreviations will be used:

$$\begin{aligned} l_n &:= \sum_{i=1}^{n-1} i &&= 1+2+\ldots+(n-1) \\ a_n &:= \sum_{i=1}^{n-1} i^2 &&= 1^2+2^2+\ldots+(n-1)^2 \\ \tau_n &:= na_n - l_n^2 \end{aligned}$$

If we denote the unknown square sides by x_i as before, and the now unknown increment (formerly b) by y, the given set of equations translates into modern notation as

$$\sum_{i=1}^{n} x_i^2 = A \tag{17}$$

$$\sum_{i=1}^{n} x_i = \Lambda \tag{18}$$

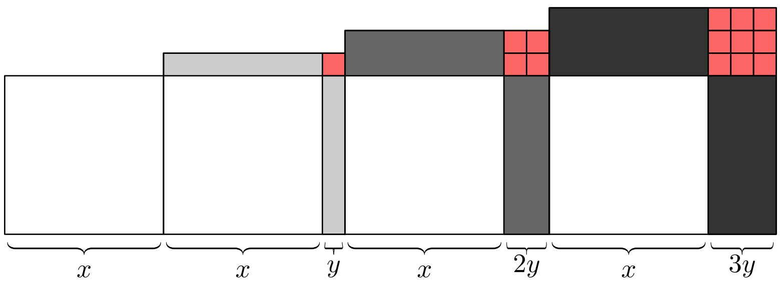
$$x_i = x_{i-1} + y \qquad \text{for} \quad 2 \leq i \leq n \tag{19}$$

where the total length Λ and the total area A are given (as numbers). This means in particular that

$$\Lambda \;=\; nx_1 + \left(\sum_{i=1}^{n-1} i\right) y \;=\; nx_1 + l_n y\,. \tag{20}$$

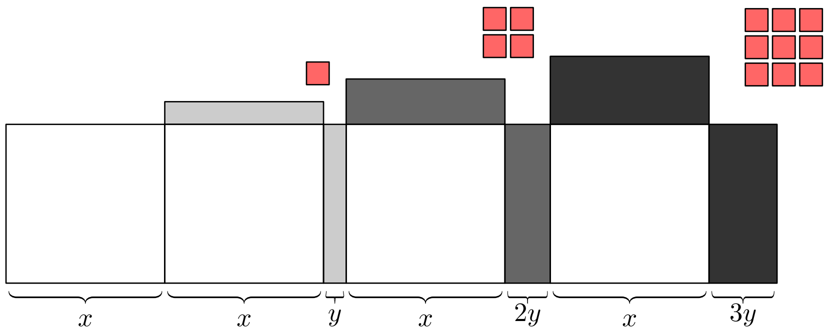
Note that the additional equation (18), as compared to type Ib, is necessary to make up for the fact that there is one additional unknown (namely y).

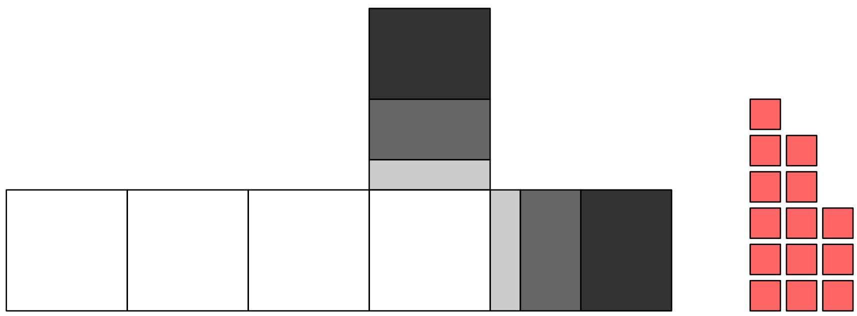
For $n = 4$ the total array formed by the four unknown squares is then depicted as follows (writing x instead of x_1):



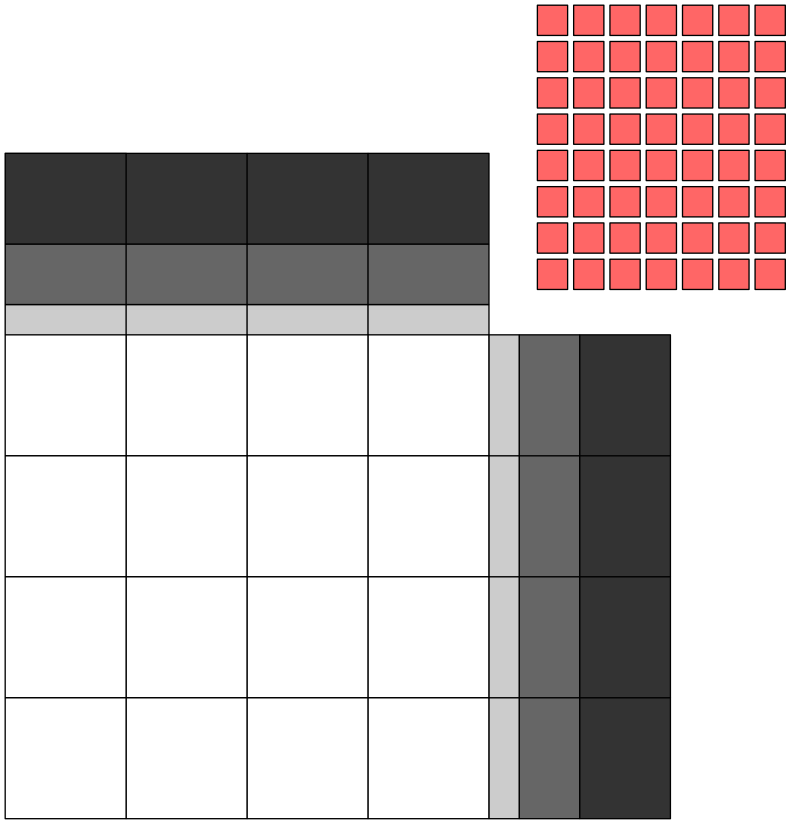
Its area is A and its width is Λ. (For $n = 4$ we have $\Lambda = 4x + y + 2y + 3y$.)

Unlike in the case of type Ib, here the areas of the little squares in the upper right corners of the second, third, ... $(n-1)^{\text{st}}$ unknown squares (i.e. the ones with side lengths $y, 2y, \ldots, (n-1)y$ and therefore areas $y^2, 4y^2, \ldots, (n-1)^2y^2$) cannot be subtracted from the total area A because now y is unknown. Therefore, in the drawing these little squares are not removed this time, but instead are taken aside to be dealt with later. They are stored as a bunch of $a_n = 1^2 + 2^2 + \ldots + (n-1)^2$ squares (red) with side lengths y (subsequently called "y-squares"):


x
x
y
x
2y
x
3y

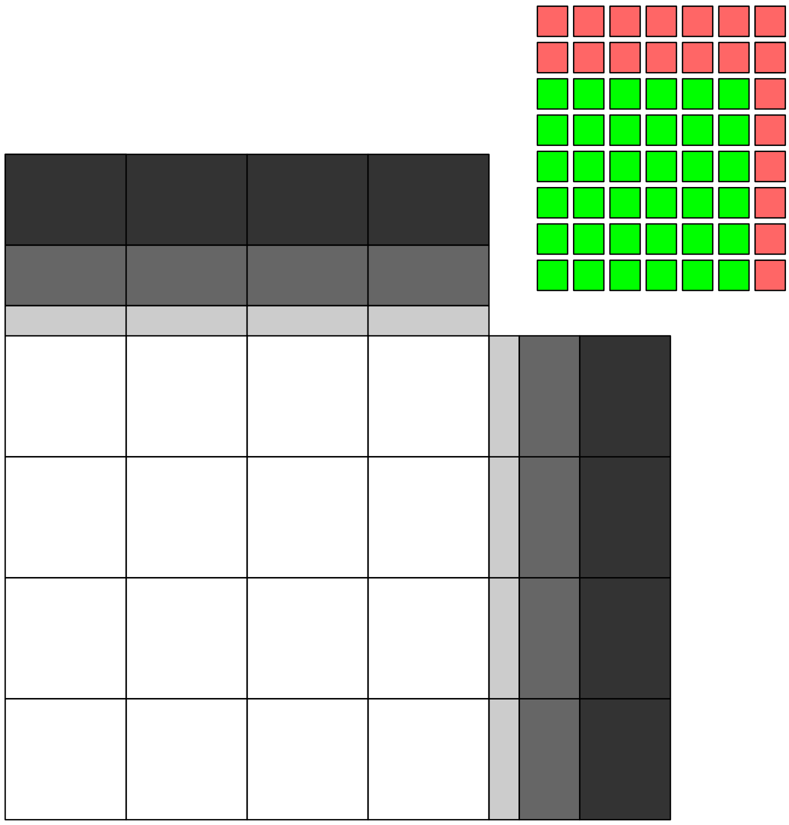


In the next step the total array (including the y-squares that have been set aside) is taken n (i.e. the number of unknown squares) times. Similar to the case of type Ib the result can be arranged in the following way:

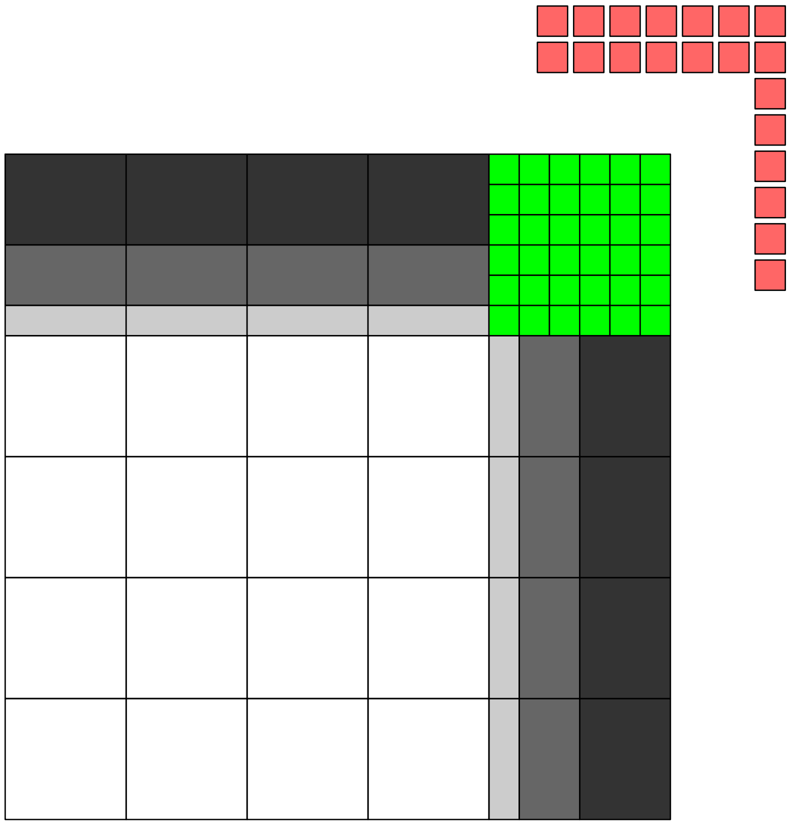


The resulting figure is a square with side length Λ, minus a square of side length $y + 2y + \ldots + (n-1)y$ in the upper right corner, plus a bunch of $na_n = n[1^2 + 2^2 + \ldots + (n-1)^2]$ (red) y-squares. Its area is nA.

The square of side length $y + 2y + \ldots + (n-1)y$ that is missing in the upper right corner can be filled up with $l_n^2 = [1 + 2 + \ldots + (n-1)]^2$ (green) out of the altogether $na_n = n[1^2 + 2^2 + \ldots + (n-1)^2]$ y-squares:



This rearranges into a complete square with side length Λ, plus a surplus of $\tau_n = na_n - l_n^2$ (red) y-squares (Remember that y is still unknown!):



The area of this surplus is obviously $\tau_n y^2$. But it is also the total area of the figure, minus the area of the complete square (with side length Λ), i.e. $nA - \Lambda^2$. Therefore one has

$$\tau_n y^2 \;=\; nA - \Lambda^2$$

and finally obtains the solution for the (previously unknown) increment:

$$y \;=\; \sqrt{\frac{1}{\tau_n}\left(nA - \Lambda^2\right)}\,. \tag{21}$$

From (20) we find

$$nx_1 = \Lambda - l_n y$$

and thus

$$\begin{aligned} x_1 \;&=\; \frac{1}{n}\left(\Lambda - l_n y\right) \\ &=\; \frac{1}{n}\left(\Lambda - l_n\sqrt{\frac{1}{\tau_n}\left(nA - \Lambda^2\right)}\right)\,. \end{aligned} \tag{22}$$

The other x_i are obtained by subsequent addition of y:

$$x_i \;=\; x_1 + (i-1)y \qquad \text{for} \quad 2 \le i \le n\,. \tag{23}$$

The necessary values for l_n and τ_n are easily computed. The following table shows the first examples:

$n = 2$	$l_2 = 1$	$\tau_2 = 1$
$n = 3$	$l_3 = 3$	$\tau_3 = 6$
$n = 4$	$l_4 = 6$	$\tau_4 = 20$
$n = 5$	$l_5 = 10$	$\tau_5 = 50$
$n = 6$	$l_6 = 15$	$\tau_6 = 105$
$n = 7$	$l_7 = 21$	$\tau_7 = 196$
$n = 8$	$l_8 = 28$	$\tau_8 = 336$

3.2 The case of negative increment

The situation is as in section 3.1, but now the side lengths of the subsequent unknown squares *decrease*. Therefore the set of equations is now

$$\sum_{i=1}^{n} x_i^2 = A \tag{24}$$

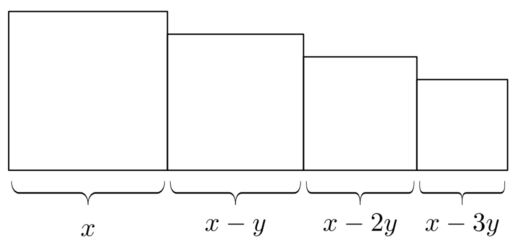
$$\sum_{i=1}^{n} x_i = \Lambda \tag{25}$$

$$x_i = x_{i-1} - y \qquad \text{for} \quad 2 \le i \le n \tag{26}$$

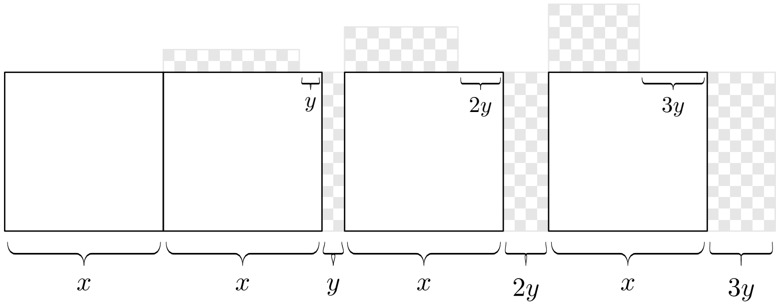
where the total length Λ and the total area A are given (as numbers) and the unknown y is supposed to be > 0. This means in particular that

$$\Lambda \;=\; nx_1 - \left(\sum_{i=1}^{n-1} i\right) y \;=\; nx_1 - l_n y\,. \tag{27}$$

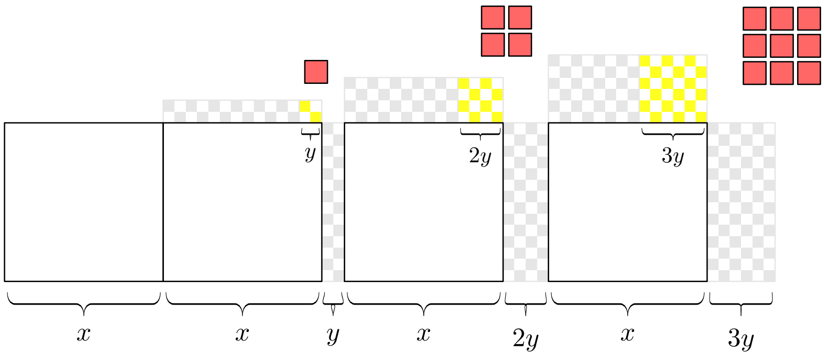
Again, while the explanations will cover the general case for arbitrary n, the drawings will depict the special situation for $n = 4$ (like in YBC 4714, no. 2). The abbreviations l_n, a_n, and τ_n from above are still in use. The conventions for the use of pixelated (i.e. "negative") rectangles are as in section 2.3. For $n = 4$ the situation is depicted by the figure (writing x instead of x_1 again)



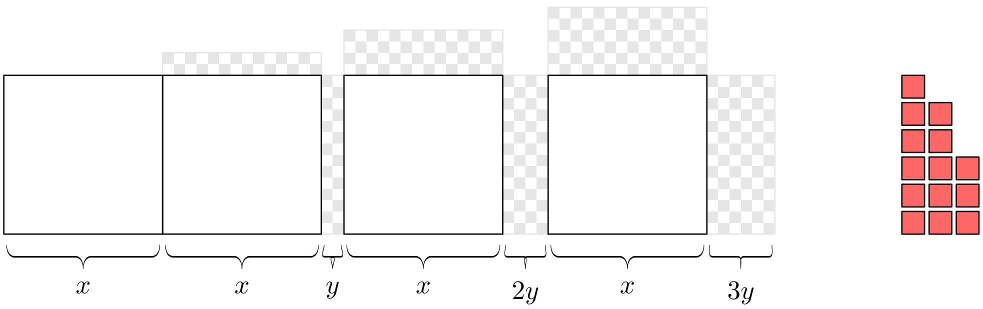
or alternatively, using pixelated rectangles, by (cf section 2.3)



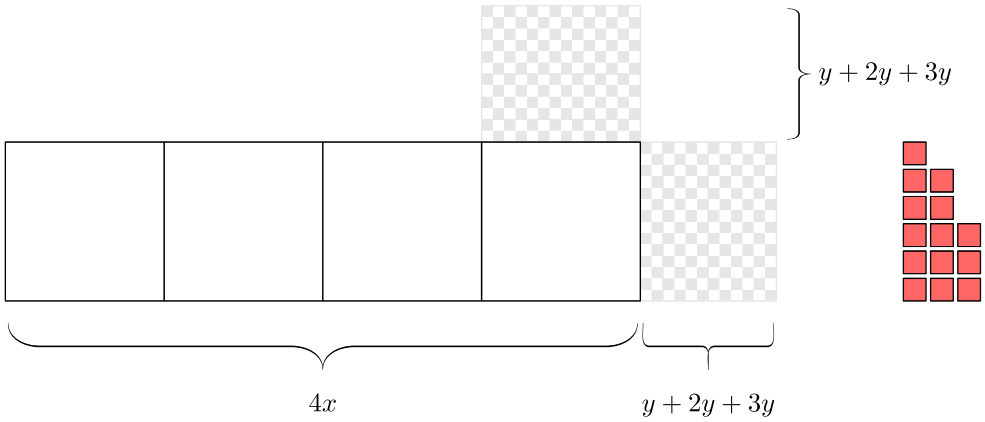
The squares with side lengths $y, 2y, \ldots, (n-1)y$ (and therefore areas $y^2, 4y^2, \ldots, (n-1)^2y^2$) in the upper right corners of the second, third, $\ldots$ $(n-1)^{\text{st}}$ unknown squares are taken out of the array (decomposed into (red) y-squares, cf section 3.1) which is realized by *adding* their respective pixelated counterparts (yellow) to the array



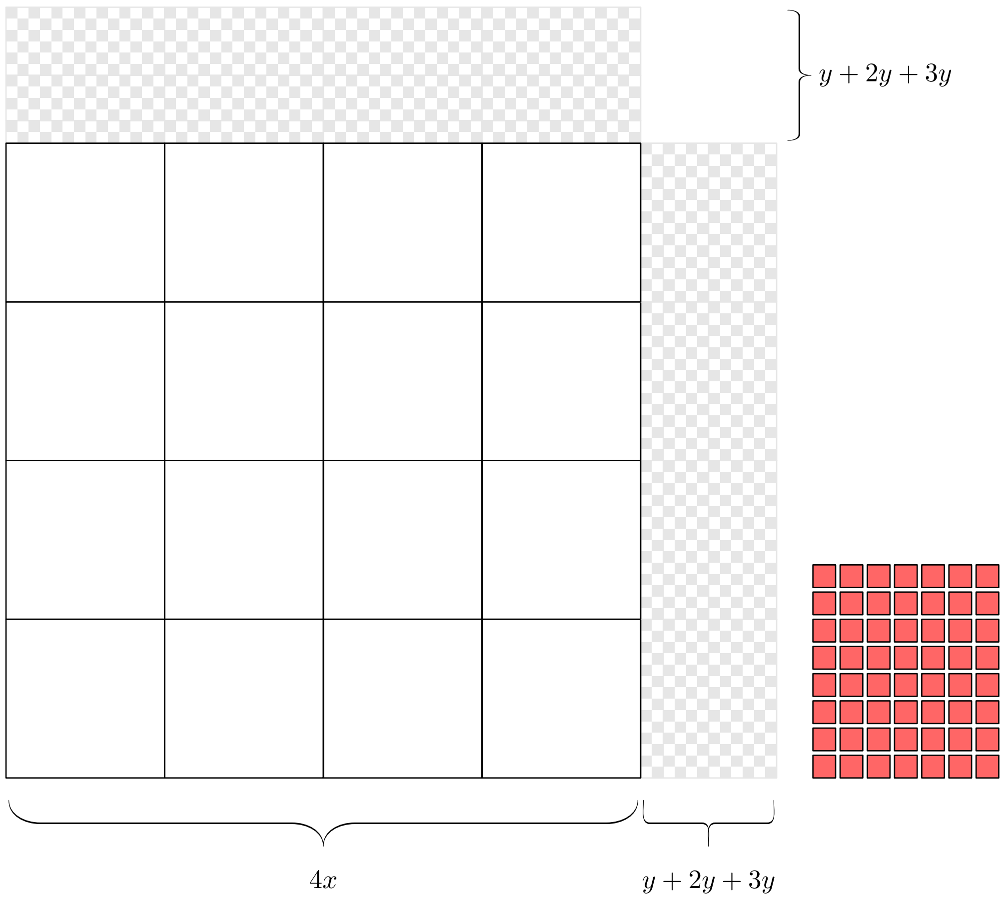
and then are stored as a bunch of $a_n = 1^2 + 2^2 + \ldots + (n-1)^2$ red y-squares on the side:



The figure is rearranged as usual

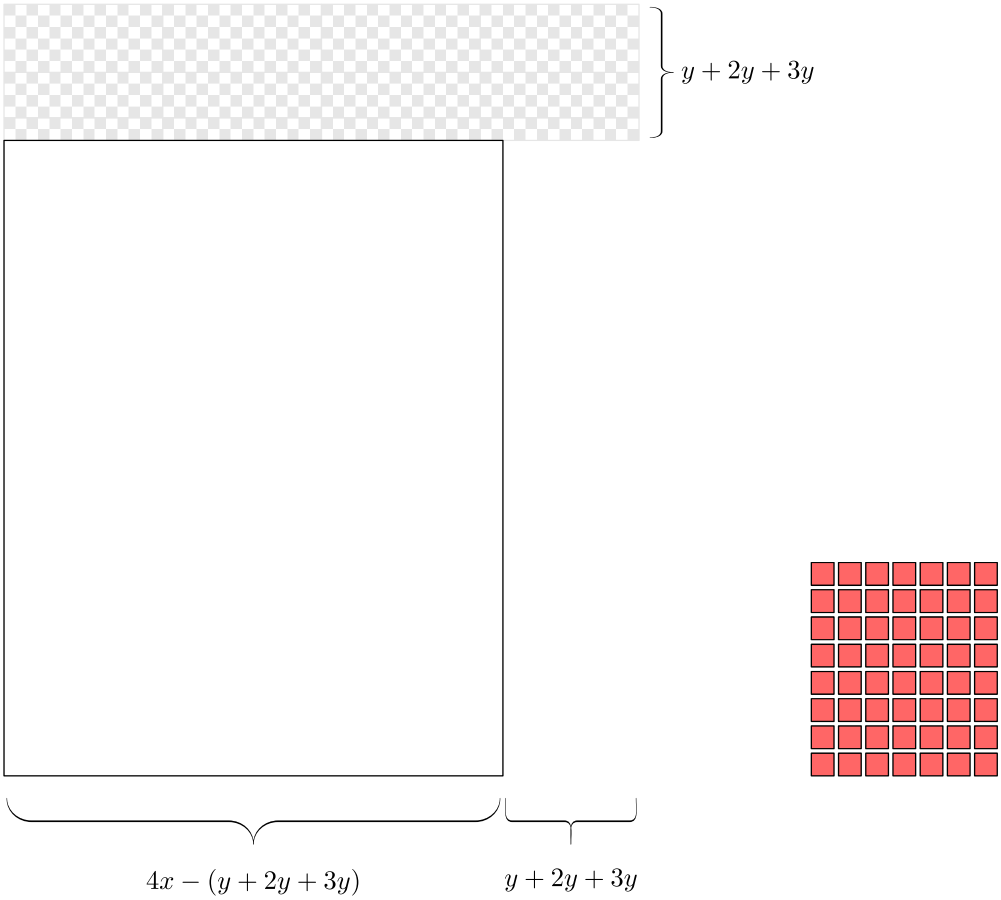


and taken n (i.e. the number of unknown squares) times:

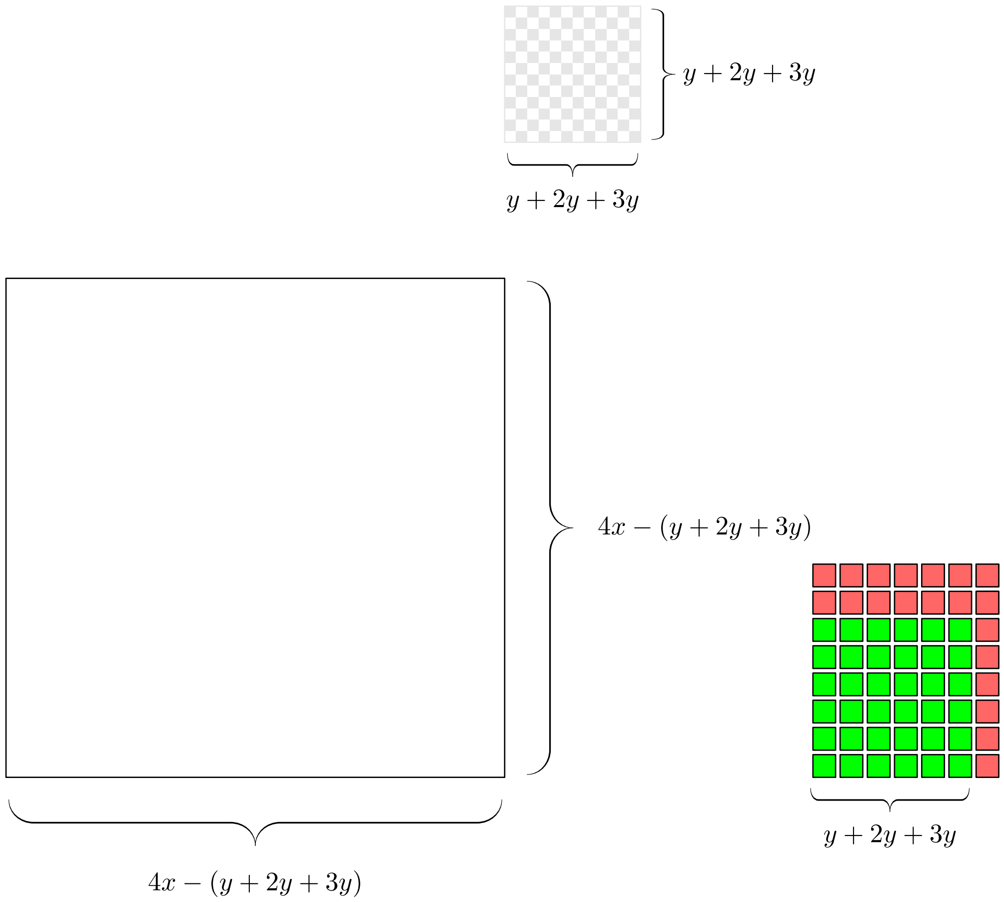


The resulting figure consists of a solid square with side length nx_1, a vertical and a horizontal pixelated (i.e. "negative") rectangle with side lengths nx_1 and $y + 2y + \ldots + (n-1)y$ each, plus the bunch of $na_n = n[1^2 + 2^2 + \ldots + (n-1)^2]$ red y-squares. Its area is nA.

Now the two pixelated rectangles (with side lengths nx_1 and $y + 2y + \ldots + (n-1)y$ each) are taken care of exactly as in the situation in section 2.3: First, the vertical one is subtracted from the solid square

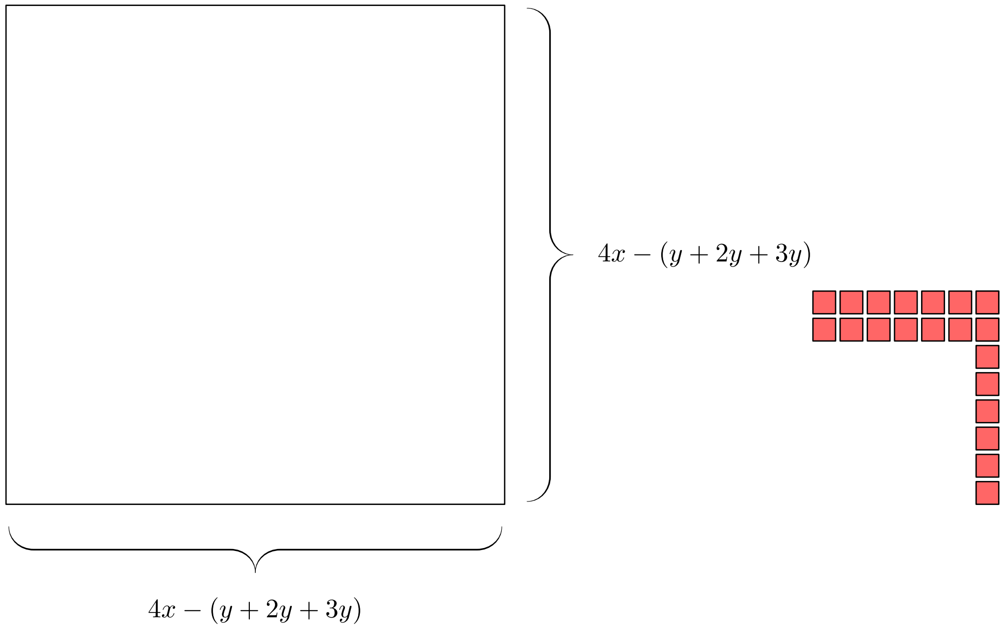


and then the horizontal one:



This leave us with a solid square with side length $nx_1 - (y + 2y + \ldots + (n-1)y) = nx_1 - l_n y = \Lambda$ (cf (27)), a pixelated square with side length $y + 2y + \ldots + (n-1)y = l_n y$, and the bunch of na_n colored y-squares. The total area is still nA.

The pixelated square with side length $l_n y = y + 2y + \ldots + (n-1)y$ eliminates $l_n^2 = [1 + 2 + \ldots + (n-1)]^2$ of the na_n colored y-squares, namely the green ones.



The resulting figure (with total area still nA) consists of the solid square with side length Λ and $\tau_n = na_n - l_n^2$ red y-squares. Therefore, $nA = \Lambda^2 + \tau_n y^2$ from which follows

$$y = \sqrt{\frac{1}{\tau_n}\left(nA - \Lambda^2\right)}, \tag{28}$$

exactly as in section 3.1. Because of (27) we have

$$nx_1 = \Lambda + l_n y$$

and thus

$$\begin{aligned} x_1 &= \frac{1}{n}\left(\Lambda + l_n y\right) \\ &= \frac{1}{n}\left(\Lambda + l_n\sqrt{\frac{1}{\tau_n}\left(nA - \Lambda^2\right)}\right). \end{aligned} \tag{29}$$

The other x_i are obtained by subsequent subtraction of y:

$$x_i = x_1 - (i-1)y \qquad \text{for} \quad 2 \le i \le n. \tag{30}$$

4 The Type Ic

This time, in order to illuminate on a fundamental level how the solution procedure can be viewed as a combination of those for types Ia and Ib, let us start with the general case for n variables and go into specific examples afterwards.

4.1 The General Case

4.1.1 The Setting

The general set of equations is

$$\sum_{i=1}^{n} x_i^2 = A \tag{31}$$

$$x_i = \frac{s_i}{t_i} x_{i-1} + b_i \qquad \text{for} \quad 2 \leq i \leq n \tag{32}$$

where $s_i, t_i \in \mathbb{N}, s_i \geq 0, t_i > 0, b_i \in \mathbb{R}$ and $A > 0$ are given constants and the x_i are the unknowns asked for. (Without loss of generality, for every i, s_i and t_i can be assumed to have no common divisors, i.e. the fraction $\frac{s_i}{t_i}$ cannot be simplified.)
The linear part (32) leads to (see Fig. 1 for orientation)

$$x_i = x_i' + \Delta_i \qquad (1 \leq i \leq n) \tag{33}$$

where

$$x_1' := x_1 \tag{34}$$

$$x_i' := \left(\prod_{j=2}^{i} \frac{s_j}{t_j} \right) x_1 \qquad \text{for} \quad 2 \leq i \leq n \tag{35}$$

and the Δ_i are given by

$$\Delta_1 := 0 \tag{36}$$

$$\Delta_2 := b_2 \tag{37}$$

$$\Delta_i := \frac{s_i}{t_i} \Delta_{i-1} + b_i \qquad \text{for} \quad 3 \leq i < n \tag{38}$$

whence

$$\Delta_i = \sum_{j=3}^{i} \left(\prod_{k=j}^{i} \frac{s_k}{t_k} \right) b_{j-1} + b_i \qquad \text{for} \quad 3 \leq i \leq n\,. \tag{39}$$

Thus the i-th square (with side length x_i) is made up from a square of side length x'_i (wich will be called "the i-th *base square*", drawn in red color in Fig. 2), and (in the case $i \geq 2$) two additional rectangles with side lengths x'_i and Δ_i (light green in Fig. 2) and an additional square of side length Δ_i (intense green in Fig. 2).

4.1.2 The Solution Procedure

The solution procedure now presents itself as a combination of the procedures for types Ia and Ib as follows.

1. Like in the case of type Ib start by removing those parts of the original squares (side length x_i) that are entirely computable by given and therefore known values (here the s_i, t_i, b_i), i.e. the squares with side lengths Δ_i (Fig. 3). The area of the remaining figure is

$$A' := A - \sum_{i=2}^{n} \Delta_i^2 \tag{40}$$

$$= A - b_2^2 - \sum_{i=3}^{n} \left[\sum_{j=3}^{i} \left(\prod_{k=j}^{i} \frac{s_k}{t_k} \right) b_{j-1} + b_i \right]^2 . \tag{41}$$

2. Like in the case of type Ia, the base squares are subdivided into little squares of side length $u = \left(\prod_{i=2}^{n} \frac{1}{t_i} \right) x_1$ and area $S = \left[\left(\prod_{i=2}^{n} \frac{1}{t_i} \right) x_1 \right]^2$. See Fig. 4. This length u fits into x'_i, the side length of the i-th base square, p_i times. The p_i obviously satisfy

$$p_1 = \prod_{j=2}^{n} t_j \tag{42}$$

$$p_i = \frac{s_i}{t_i} p_{i-1} \qquad \text{for} \quad 2 \leq i \leq n \tag{43}$$

and therefore

$$p_i = \left(\prod_{j=2}^{i} s_j \right) \left(\prod_{k=i+1}^{n} t_k \right) \qquad \text{for} \quad 1 \leq i \leq n. \tag{44}$$

The total amount Q of little squares is therefore

$$Q = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} \left[\left(\prod_{j=2}^{i} s_j \right) \left(\prod_{k=i+1}^{n} t_k \right) \right]^2 . \tag{45}$$

3. Additionally every rectangle with side lengths x_i' and Δ_i $(i \geq 2)$ is decomposed into p_i stripes each of which has length Δ_i and width u (Fig. 4). Note that $u = \frac{1}{p_i} x_i'$ for all i.

4. Rearrange the remaining figure by arranging all the Q little squares into a (horizontal) row and combining all the horizontal stripes into one long horizontal super-stripe and all the vertical stripes into one long vertical super-stripe. The width of each of the two super-stripes is u. The length L of each super-stripe is

$$L = \sum_{i=2}^{n} p_i \Delta_i \,.$$

Observe that step 4 transforms the situation into the one from type Ib: There are now a number (namely Q) of equally large squares and two rectangular attachments whose widths equal the side length of these squares. The subsequent steps are completely analogous to what happens in the case of type Ib.

5. Take all this Q times. Arrange the resulting Q^2 little squares into a huge square with side length Qu. Arrange the resulting Q horizontal and vertical super-stripes on the right and atop the huge square respectively (Fig. 5). The new figure has the area

$$\begin{aligned} A'' &:= QA' \\ &= Q\left(A - \sum_{i=2}^{n} \Delta_i^2\right) \\ &= Q\left(A - b_2^2 - \sum_{i=3}^{n}\left[\sum_{j=3}^{i}\left(\prod_{k=j}^{i}\frac{s_k}{t_k}\right) b_{j-1} + b_i\right]^2\right) \\ &= \left(\sum_{i=1}^{n}\left[\left(\prod_{j=2}^{i} s_j\right)\left(\prod_{k=i+1}^{n} t_k\right)\right]^2\right)\left(A - b_2^2 - \sum_{i=3}^{n}\left[\sum_{j=3}^{i}\left(\prod_{k=j}^{i}\frac{s_k}{t_k}\right) b_{j-1} + b_i\right]^2\right) \end{aligned}$$

and the shape of a very huge square of side length $Qu + L$, with a square of side length L missing in the upper right corner.

6. Adding this missing square (which of course has the area L^2) one obtains the complete very huge square (Fig. 6). Its side length is $Qu + L$ and its area is $A''' := A'' + L^2$.

7. Extracting the square root of A''', the new square's area, therefore gives $Qu + L$, its side length:

$$Qu + L = \sqrt{A'''}\,.$$

8. From this square root subtract L, the length of the super-stripe. This gives Qu:

$$Qu = \sqrt{A'''} - L$$

whence $u = \frac{\sqrt{A'''} - L}{Q}$. Since $u = \left(\prod_{i=2}^{n} \frac{1}{t_i}\right) x_1$, the first side length is $x_1 = u \prod_{i=2}^{n} t_i$.

To sum it up,

$$x_1 = u \prod_{i=2}^{n} t_i \tag{46}$$

$$= \frac{\sqrt{A'''} - L}{Q} \prod_{i=2}^{n} t_i \tag{47}$$

$$= \frac{\sqrt{A'' + L^2} - L}{Q} \prod_{i=2}^{n} t_i \tag{48}$$

$$= \frac{\sqrt{QA' + L^2} - L}{Q} \prod_{i=2}^{n} t_i \,. \tag{49}$$

The remaining x_i are computed my means of (32).

Remark 4.1 Remark 1.2 applies accordingly.

Remark 4.2 If one (artificially) replaces all the p_i with qp_i where q is a positive integer > 1, i.e. multiplies the number of little squares by q^2 and the number of (super-)stripes by q, one just scales the problem up. The result is naturally the same. Cf Remark 1.3. (This is actually done in the cuneiform examples below.)

Remark 4.3 Of course, if $b_i = 0$ for all i, the system reduces to Type Ia. And if $\frac{s_i}{t_i} = 1$ for all i it reduces to Type Ib. If s_i and t_i are taken to be both 1 also the described method reduces to the one used in the cuneiform text in the case $n = 3$ (BM 13901, no. 18; see section 2.1). It also works perfectly for the case $n = 2$, but here the cuneiform example (BM 13901, no. 9; see Appendix A) uses a different approach.

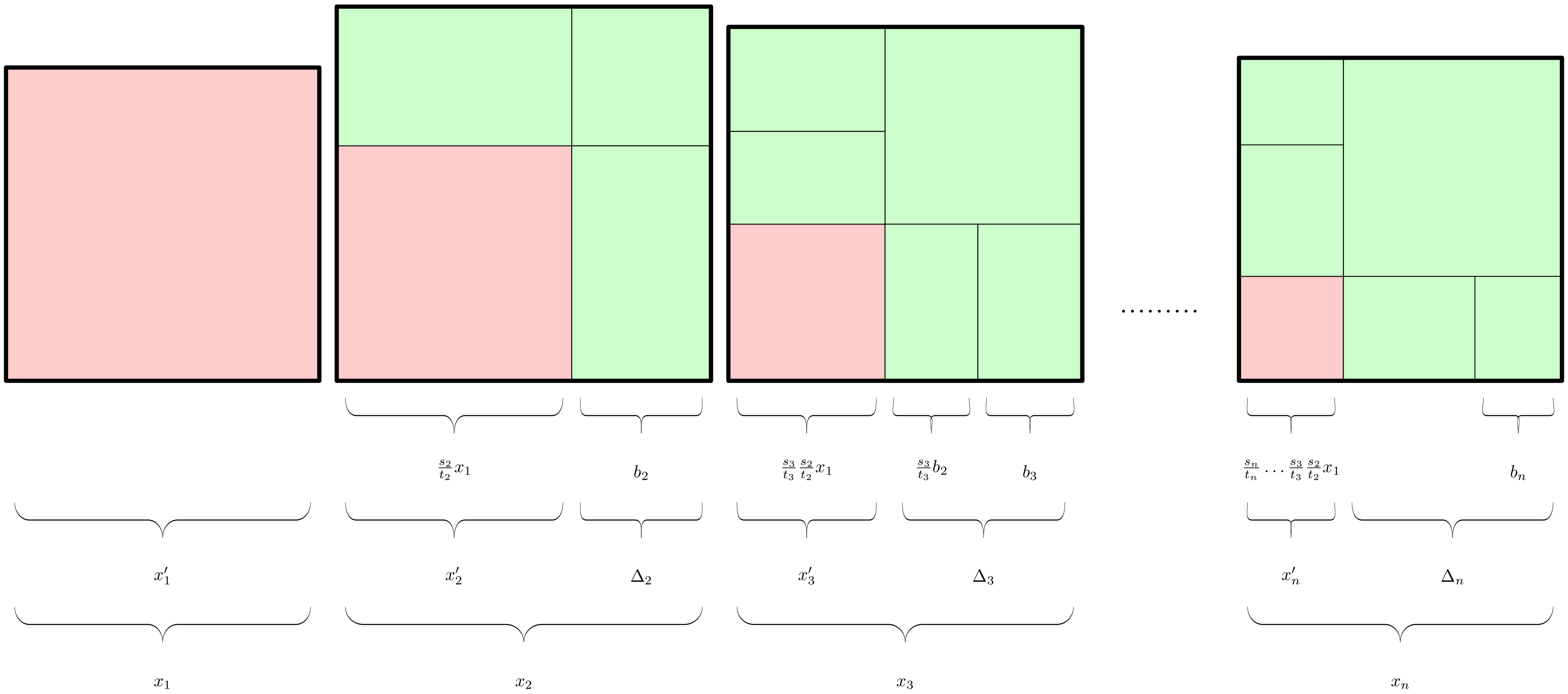


Figure 1: Type Ic: The setting

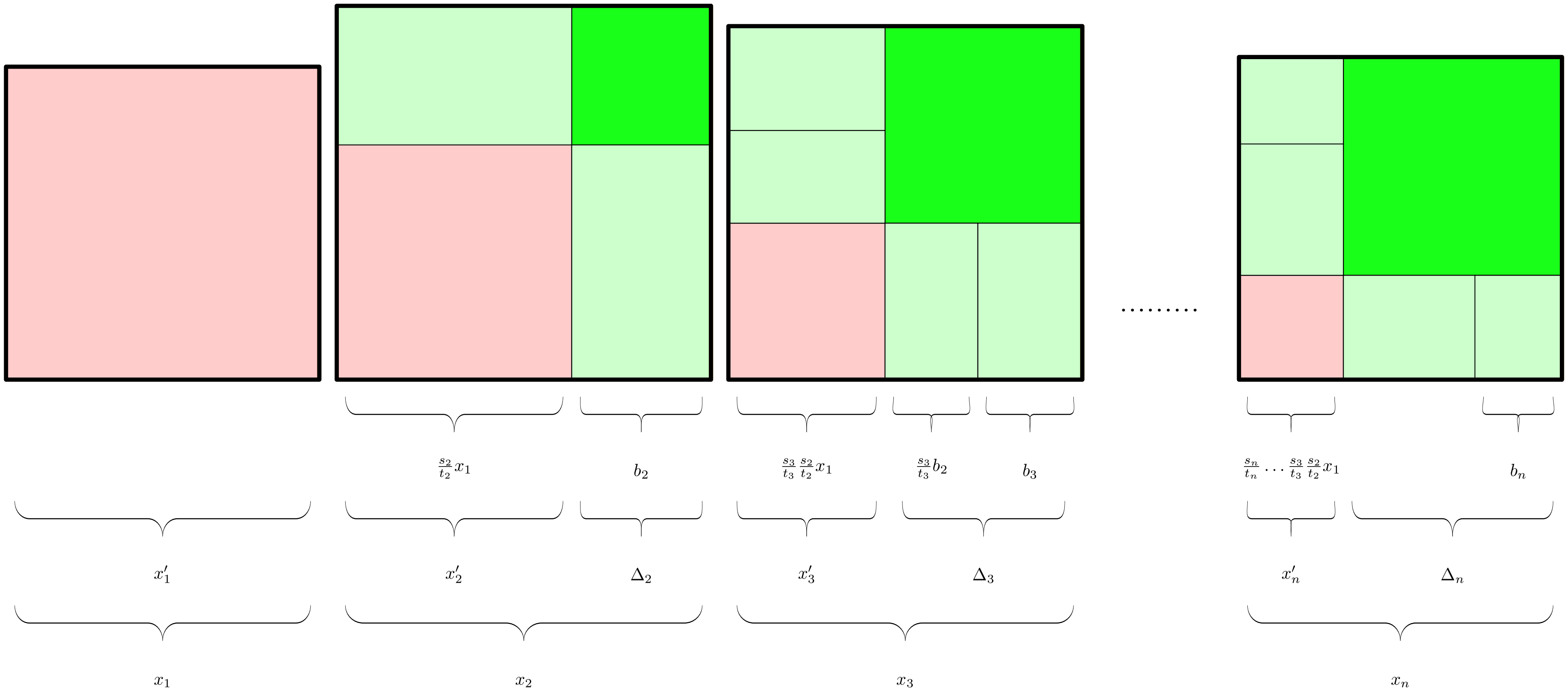


Figure 2: Type Ic: Identifying the constants

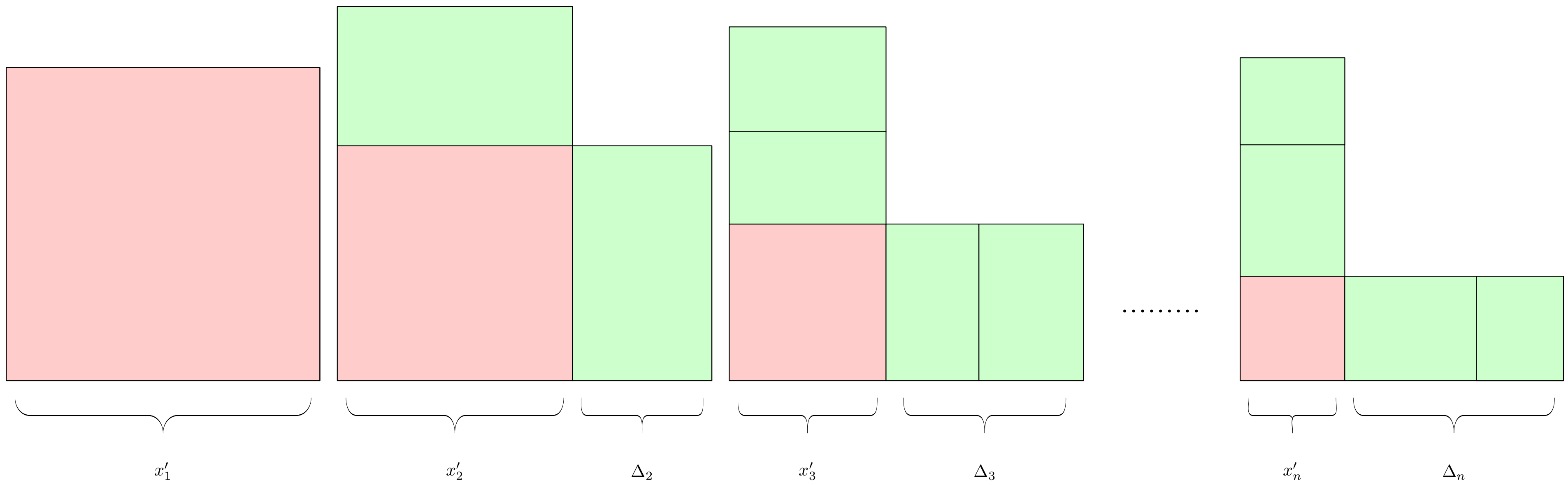


Figure 3: Type Ic: Removing the squares with side lengths Δ_i

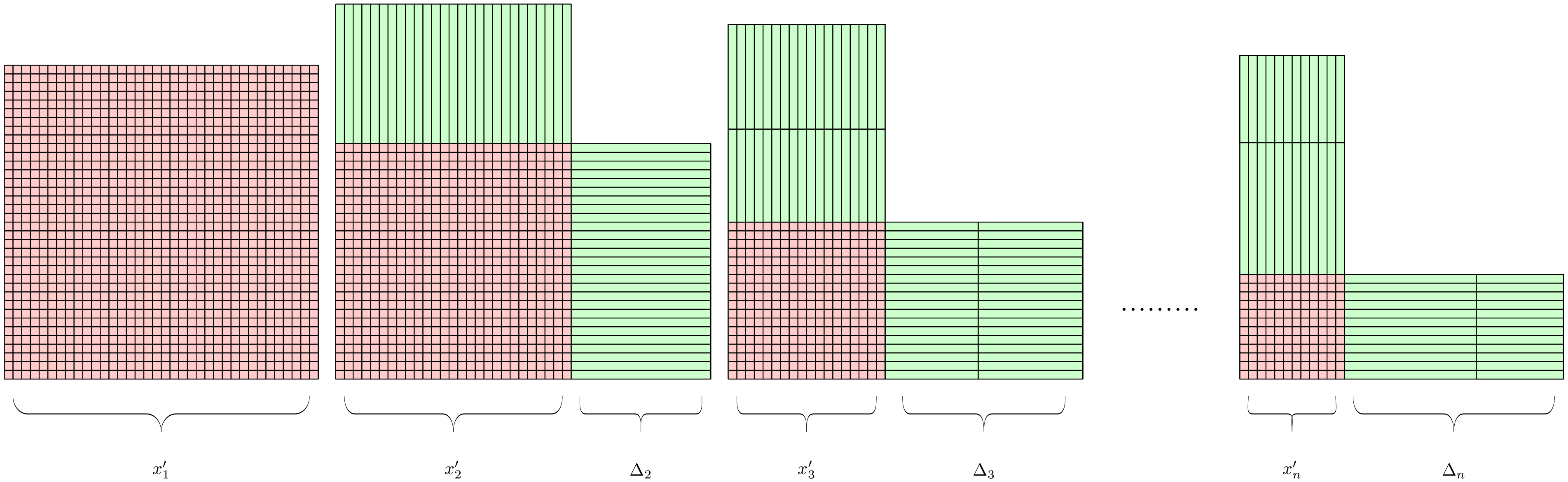


Figure 4: Type Ic: Subdividing the remaining squares and rectangles

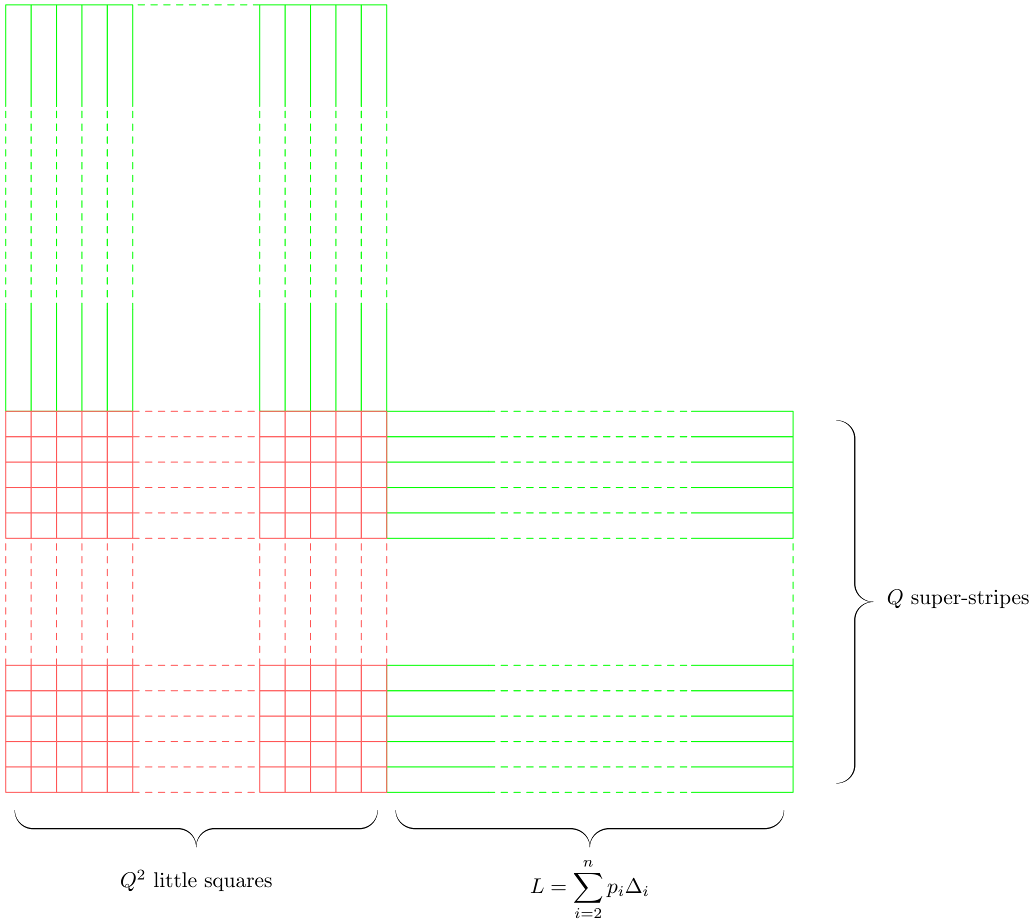


Figure 5: Type Ic: The super-construction

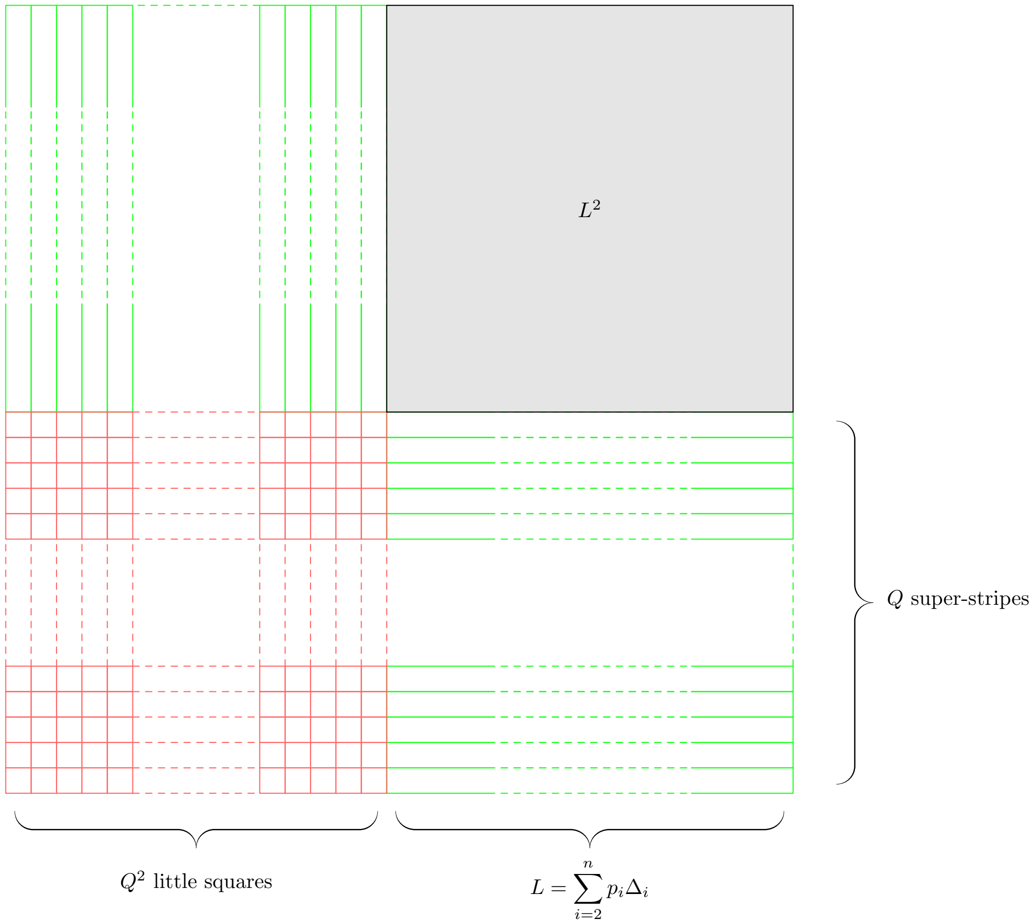
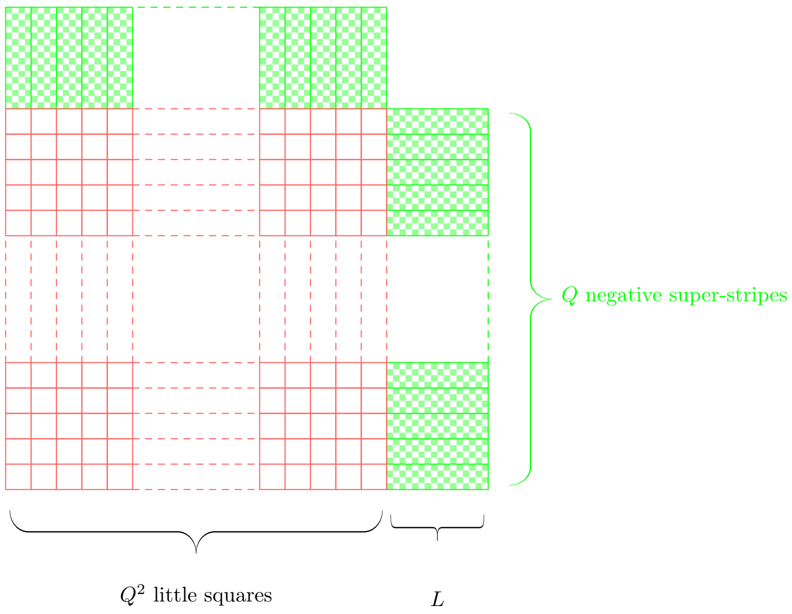


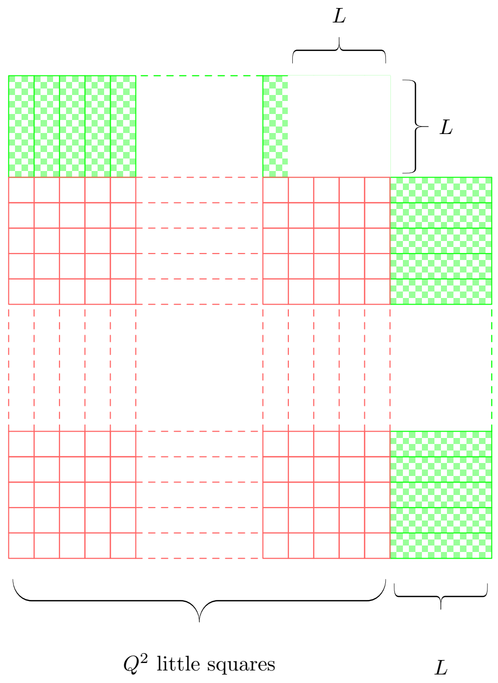
Figure 6: Type Ic: Completing the Square

Remark 4.4 A word is in order concerning the situation that magnitudes, e.g. b_i (or in the next section, γ_i, etc.), "are negative" and what it means to say that the procedure also works in these cases. Of course these magnitudes represent geometric properties like length or area and can as such not be "negative". Also the Mesopotamian mathematicians didn't deal with negative numbers in the modern sense, but considered the corresponding ("positive") magnitude as subtracted or to be subtracted. It is only the modern algebraic formalism that allows us today to express a "magnitude to be subtracted" as a negative number. Admittedly this obscures much of the geometric nature of our procedures but, on the other hand, has the advantage of a compact notation that covers different cases by only one expression. For example, if one problem states that "one square side exceeds the other by b" and another one states that "one square side is reduced with respect to the other by b" we can either bother to write it as $x_2 = x_1 + b$ and $x_2 = x_1 - b$, respectively, bearing in mind that b is naturally positive. Or we can just write $x_2 = x_1 + b$ by allowing for b to be negative.

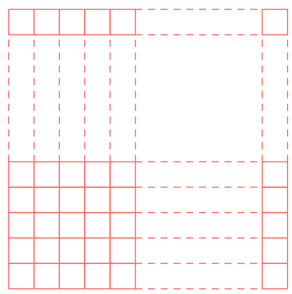
This being said, the point is the following. It can happen that the magnitudes (e.g. b_i) that are subtracted exceed the ones that are added so that in the end (the area of) our super-stripe is to be subtracted from (the area of) the Q^2 little squares instead of added to them. This situation can be dealt with graphically by using pixelated rectangles to symbolize magnitudes-to-be-removed, as introduced in section 2.3. The entity with area A'' is now represented by the figure



(which replaces Fig. 5). Adding L^2 (step 6 in the procedure) amounts to *eliminating* a square with side length L from the pixelated (negative) green area which leads to



(which therefore replaces Fig.6). Taking proper care of the pixelated areas (recall section 2.3) yields a complete square with side length $Qu - L$:



This is the entity with area A'''. So in order to obtain Qu (from which then x_1 is found, step 8 above) L must now be *added* to $\sqrt{A'''}$ instead of subtracted from it. This is the only change in the solution procedure.

In the formal notation, however, the very reasons that cause "the super-stripe to be subtracted" give rise to a negative value for L. And this must still be *subtracted* from $\sqrt{A'''}$ in order to add its absolute value. Therefore, the change in the procedure described above is invisible in its formal description.

4.2 The Case of Two Variables

The general situation described in section 4.1 translates into the case for two variables as follows. Note that in the example drawings below the constants are chosen such that $s < t$ and $b > 0$. Of course the method works as well if $s > t$ (cf the example BM 13901, no. 11 in section 1.1) and even for $b < 0$ (see Remark 4.4 on page 54 and example Strssbg. 363, obv. 1-12 in 4.2.2 below). The same holds for the case of three variables in section 4.3 below. In the following descriptions, for every single step of the solution procedure reference to the corresponding step in the cuneiform examples below (A to O, respectively A to T) is given in blue colour.

Given is a system of two equations in two variables x and y:

$$\begin{aligned} x^2 + y^2 &= A\,, && \text{(stated in step A)} \\ y &= \tfrac{s}{t}x + b\,, && \text{(stated in step B)} \end{aligned} \qquad (50)$$

where t and s are positive integers and b is an arbitrary number (Fig. 7). All of them are "written down" in step C, in this order, a step which is missing in Strssbg. 363, obv. 1-12, though. (The numbers t and s are just p_1 and p_2 in the general setting of section 4.1.)

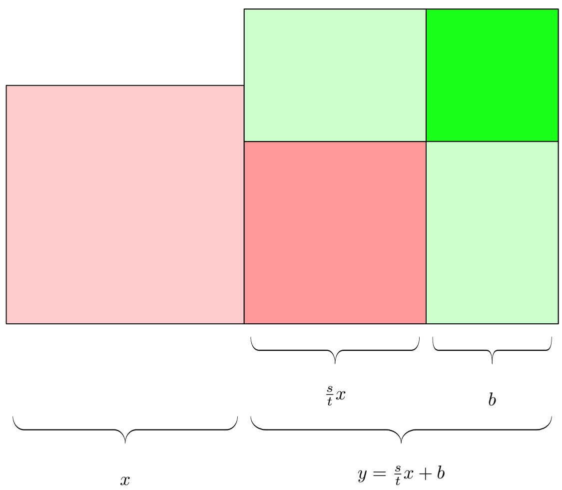


Figure 7

The first step is to cut the intense green square with side length b out of the total figure (Fig. 8). This can be done because its size is entirely known since it does not contain any variable and so its area b^2 is easily found (step D).

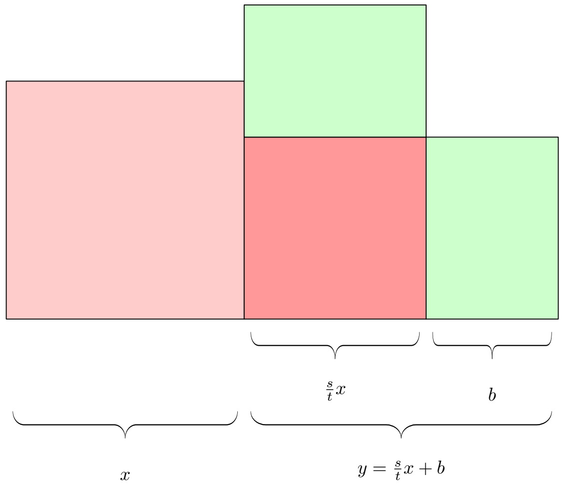


Figure 8

The area of the truncated figure is therefore

$$A' := A - b^2 \,. \qquad \text{(step E)}$$

Then the left red square of the remaining shape is subdivided into t^2 (computed in step F_1) little squares with side lenght $\frac{x}{t}$, the right red square into s^2 (computed in step F_2) such little squares. Also, each of the two green rectangles is split into s stripes of width $\frac{x}{t}$ and length b (Fig. 9).

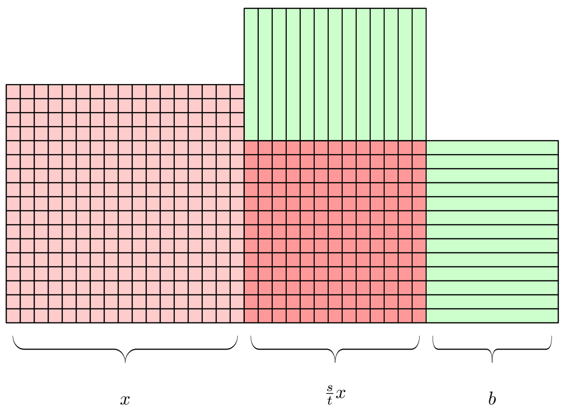


Figure 9

So the truncated figure consists of altogether $[t^2 + s^2]$ (computed in step G) little squares (with side lenght $\frac{x}{t}$) and s horizontal and s vertical rectangular stripes (of width $\frac{x}{t}$ and length b) (Fig. 10).

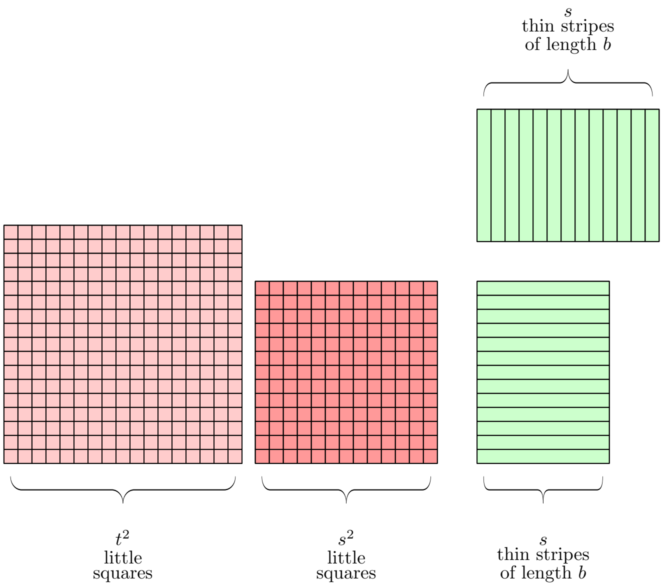


Figure 10

Next imagine all the horizontal stripes arranged into one long horizontal super-stripe with width $\frac{x}{t}$ and length

$$L := sb\,, \qquad \text{(computed in step I)}$$

and all the vertical stripes into one vertical super-stripe of the same dimensions. So far we have only changed the shape of the figure, but not its area. The area of the new arrangement is still A'.

Now the whole arrangement is taken $[t^2 + s^2]$ (which is the total number of little squares) times. The resulting entity has now the total area

$$A'' := [t^2 + s^2]A' \qquad \text{(computed in step H)}$$

and consists of $\left[t^2 + s^2\right]^2$ little squares and $[t^2 + s^2]$ horizontal and $[t^2 + s^2]$ vertical super-stripes. Imagine them arranged in the following way: The $\left[t^2 + s^2\right]^2$ little squares are made to constitue a super-square, each side of which consists of $[t^2+s^2]$ little squares (of side lenght $\frac{x}{t}$). So the side lengths of the super-square is $[t^2 + s^2]\frac{x}{t}$. Now, to each of the $[t^2 + s^2]$ little squares on the right edge of the super-square is attached one of the (altogether $[t^2 + s^2]$) horizontal super-stripes, and to each of the $[t^2 + s^2]$ little squares on the top edge of the super-square is attached one of the (again: altogether $[t^2 + s^2]$) vertical super-stripes (Fig. 11).

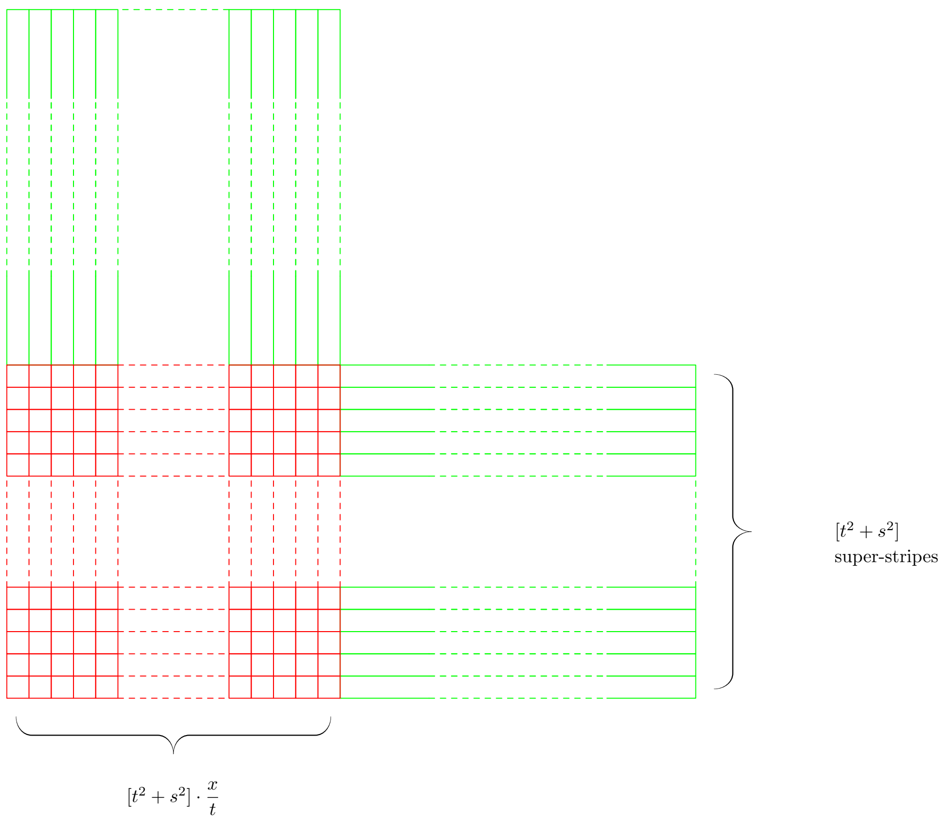


Figure 11

The resulting figure fails to be a square itself by means of a square of side length L and thus area L^2 (computed in step J). Adding this missing square we obtain a complete square with total area $A''+L^2$ (computed in step K) and side length $[t^2+s^2]\frac{x}{t}+L$. But the side length of a square is just the square root of its area, so we have $[t^2+s^2]\frac{x}{t}+L=\sqrt{A''+L^2}$ (this $\sqrt{A''+L^2}$ is computed in step L), and therefore

$$\begin{aligned}
[t^2+s^2]\tfrac{x}{t} &= \sqrt{A''+L^2}-L && \text{(step M)}\\
\tfrac{x}{t} &= \tfrac{1}{[t^2+s^2]}\left(\sqrt{A''+L^2}-L\right) && \text{(step N)} \qquad (51)\\
x &= \tfrac{t}{[t^2+s^2]}\left(\sqrt{A''+L^2}-L\right). && \text{(step O}_1\text{)}
\end{aligned}$$

Since $y=\frac{s}{t}x+b$ we have

$$y \;=\frac{s}{[t^2+s^2]}\left(\sqrt{A''+L^2}-L\right)+b \qquad (52)$$

where $\dfrac{s}{[t^2+s^2]}\left(\sqrt{A''+L^2}-L\right)$ is computed in step O_2 and finally b is added (respectively $|b|$ subtracted in Strssbg. 363, obv. 1-12 where b is "negative") in step O_3.

Resubstituting the expressions for A'' and L one obtains

$$x\;=\frac{t}{t^2+s^2}\left(\sqrt{(t^2+s^2)\,(A-b^2)+(sb)^2}-sb\right)$$

and

$$y = \frac{s}{t^2 + s^2}\left(\sqrt{(t^2 + s^2)\,(A - b^2) + (sb)^2} - sb\right) + b\,.$$

Remark 4.5 Note that in the following two examples what is written down in step C and processed in the subsequent steps, is 01 00 and 40 instead of the minimal possible values for t and s which are 3 and 2. See Remark 1.3 for that. It is also possible, however, that these numbers are to be interpreted as 01 and 00▲40, and that the solution procedure followed a somewhat more direct approach that would be a natural generalization of what has been suggested in remark 1.5. (In fact, this is exactly how Høyrup (2002, 73-77) analyses example BM 13901 no. 14 below.) This would consequently involve a multiplication by $[1^2 + (\frac{s}{t})^2]$ (which is $[1^2 + (\frac{2}{3})^2]$ in the two examples below) in the process. Recall (cf remark 1.5) that — in the special setting of Babylonian mathematics — this would not be possible if $[1^2 + (\frac{s}{t})^2]$ were sexagesimally non-regular and therefore couldn't be written as a sexagesimal fraction. For type Ic examples (for $n = 3$ and $n = 4$) where exactly this is the case see remark 4.6.

4.2.1 Example: BM 13901, no. 14

$$\begin{aligned} x^2 + y^2 &= 25\ 25 \\ y &= \tfrac{2}{3}x + 05 \end{aligned}$$

The first example is BM 13901, no. 14 (obv. ii 44 - rev. i 11).[5] Translation after Neugebauer (1937, 3). The part written in grey is destroyed on the tablet. Neugebauer's completion is based on Strssbg. 363, obv. 1-12 (see below) and the wording in the other problems on the tablet, especially no. 24 (see below).[6]

obv. ii 44) **a-ša3** *ši-ta mi-it-ha-ra-ti-ia ak-mur-ma* ⌜25] 25
45) *mi-it-har-tum ši-ni-pa-at mi-it-har-tim* [*u3* 05 **g̃a**]**r**
46) 01 *u3* 40 *u3* 05 [*e-le-nu* 4]0 *ta-la-pa-at*
47) 05 *u3* 05 [*tu-uš-ta-kal* 25 *lib3-ba* 25 25 *ta-na-sa3-ah-ma*]
rev. i 1) [25 *ta-la-pa-at* 01 *u3* 01 *tu-uš-ta-kal* 01 40 *u3* 40 *tu-uš-ta-kal*]
2) [26 40 *a-na* 01 *tu-ṣa-ab-ma* 01 26 40 *a-na* 25 *ta-na-ši-ma*]
3) [36 06 40 *ta-la-pa-at* 05 *a-na* 40 *ta-na-ši-ma* 03 20]
4) [*u3* 03 20 *tu-uš-ta-kal* 11 06 40] *a-na* 3[6] 06 40 [*tu-ṣa-ab-ma*]
5) [36 17 46 40-**e** 46 40 **ib2-sa2** 03] 20 *ša tu-uš-ta-ki*[*-lu*]
6) [*lib3-ba* 46 40 *ta-na-sa3-ah*]*-ma* 43 20 *ta-la-pa-a*[*t*]
7) [**igi** 01 26 40 *u2-la ip-pa-ṭ*]*a-ar mi-nam a-na* 01 2[6 4]0
8) [*lu-uš-ku-un ša* 43 20 *i-n*]*a-di-nam* 30 *ba-an-da-šu*
9) [30 *a-na* 01 *ta-na-ši-ma* 30] *mi-it-har-tum iš-ti-a-at*
10) [30 *a-na* 40 *ta-na-ši-ma* 20] *u3* 05 *tu-ṣa-ab-ma*
11) [25 *mi-it-har-t*]*um ša-ni-tum*

[5] Cf Høyrup (2002, 73-77).

[6] In rev. i 3) it is probably "40 *a-na* 05 *ta-na-ši-ma*" instead of "05 *a-na* 40 *ta-na-ši-ma*" (by analogy to the two other texts).

(A) I have added the areas of my two squares and 25 25 (is the result).
(B) (The second) square side is two thirds of (the preceeding) square side plus 05 added.
(C) You write down 01 00 and 40, and 05 above (the) 40.
(D) You multiply (the) 05 (given in line B) and (the) 05 (given in line B) (and the result is 25).
(E) You tear (this) 25 out of the 25 25 (given in line A) and write down (the resulting) 25 00.
(F_1) You multiply (the) 01 00 and (the) 01 00 (written down in line C) (and the result is) 01 00 00.
(F_2) You multiply (the) 40 and (the) 40 (written down in line C) (and the result is 26 40).
(G) You add (the) 26 40 (obtained in step F_2) and (the) 01 00 00 (obtained in step F_1) and (the result is 01 26 40).
(H) You multiply (the) 01 26 40 (from step G) by (the) 25 00 (obtained in step E) and write down (the resulting) 36 06 40 00.
(I) You multiply (the) 40 (written down in line C) by (the) 05 (given in line B) (and the result is 03 20).
(J) You multiply (the) 03 20 (obtained in step I) and (the) 03 20 (obtained in step I) (and the result is 11 06 40).
(K) You add (the) 11 06 40 (obtained in step J) to (the) 36 06 40 00 (obtained in step H) and (36 17 46 40 is the result).
(L) The square root of (the) 36 17 46 40 (from step K) is 46 40.
(M) You tear (the) 03 20 that you have multiplied (with itself in step J) out of (the) 46 40 (obtained in step L) and you write down (the resulting) 43 20.
(N) The inverse of (the) 01 26 40 (obtained in step G) cannot be solved. What shall I multiply to 01 26 40 that gives (the) 43 20 (obtained in step M)? Its quotient (i.e. the answer) is 00 ▲ 30.
(O_1) You multiply (the) 00 ▲ 30 (from step N) by (the) 01 00 (written down in line C) and (the resulting) 30 is the first square side.
(O_2) You multiply (the) 00 ▲ 30 (from step N) by (the) 40 (written down in line C) and
(O_3) You add (the resulting) 20 and (the) 05 (given in line B), and (the resulting) 25 is the second square side.

4.2.2 Example: Strssbg. 363, obv. 1-12

$$\begin{aligned} x^2 + y^2 &= 16\ 40 \\ y &= \frac{2}{3}x - 10 \end{aligned}$$

Transliteration of Strssbg. 363, obv. 1-12, after Neugebauer (1935, 244).

1) **a$_3$-ša** 02 **ib$_2$-sa$_2$ g̃ar-g̃ar**-*ma* 16 40 **ib$_2$-sa$_2$** $\frac{2}{3}$ **ib$_2$-sa$_2$**
2) 10 *i-na* **ib$_2$-sa$_2$ tur ba-zi ib$_2$-sa$_2$ en-nam za-e aka-da-zu-de$_3$**
3) 10 **zur-zur** 01 40 **in-šum$_2$** 01 40 *i-na* 16 40 **zi**-*ma*
4) 15 **in-šum$_2$** 01 **zur-zur**-*ma* 01 **in-šum$_2$** 40 **zur-zur**-*ma* 26 40
5) 01 **u$_3$** 26 40 **g̃ar-g̃ar**-*ma* 01 26 40 **in-šum$_2$** 01 26 40 *a-na* 15 **nim**
6) 21 40 **in-šum$_2$** 40 *a-na* 10 **nim**-*ma* 06 40 **in-šum$_2$**
7) 06 40 **zur-zur**-*ma* 44 26 40 **in-šum$_2$** 44 26 40

8) *a-na* 21 40 **tah**-*ma* 22 24 26 40 **in-šum₂**
9) 22 24 26 40**-e** 36 40 **ib₂-sa₂** 06 40 *ša tu-uš-ta-ki-lu*
10) *a-na* 36 40 **tah**-*ma* 43 20 **in-šum₂** *mi-nam a-na* 01 26 40 **he₂-g̃ar**
11) *ša* 43 20 **in-šum₂** 30 *šu-ku-un* 30 *a-na* 01 **nim**-*ma* 30 **ib₂-sa₂ gu-la**
12) 30 *a-na* 40 **nim**-*ma* 20 **in-šum₂** 10 *i-na* 20 **zi**-*ma* 10 **ib₂-sa₂ tur-ra**

(A) The areas of 2 squares added and 16 40 (is the result).
(B) (The small) square side (is) $\frac{2}{3}$ (of the big) square side (and also) 10 is subtracted in the small square side.
(–) What are the square sides? You in your doing:
(D) (The) 10 (given in line B) squared gives 01 40.
(E) (This) 01 40 subtracted from (the) 16 40 (given in line A) and it gives 15 00.
(F_1) 01 00 squared and it gives 01 00 00.
(F_2) 40 squared and (it gives) 26 40.
(G) (The) 01 00 00 (obtained in step F_1) and (the) 26 40 (obtained in step F_2) added and it gives 01 26 40.
(H) (This) 01 26 40 to (the) 15 00 (obtained in step E) multiplied gives 21 40 00 00.
(I) (The) 40 (that was squared in step F_2) to (the) 10 (given in line B) multiplied and it gives 06 40.
(J) (This) 06 40 squared and it gives 44 26 40.
(K) (This) 44 26 40 to (the) 21 40 00 00 (obtained in step H) added and it gives 22 24 26 40.
(L) The square root of (this) 22 24 26 40 is 36 40.
(M) (The) 06 40 that you have multiplied (with itself in step J) to (the) 36 40 (obtained in step L) added and it gives 43 20.
(N) What shall be multiplied to (the) 01 26 40 (obtained in step G) that gives (the) 43 20 (from step M)? Put 00 ▲ 30!
(O_1) (This) 00 ▲ 30 by (the) 01 00 (that was squared in step F_1) multiplied and (the resulting) 30 is the big square side.
(O_2) (The same) 00 ▲ 30 by (the) 40 (that was squared in step F_2) multiplied and it gives 20.
(O_3) (The) 10 (given in line B) from (the) 20 (obtained in step P) subtracted and (the resulting) 10 is the small square side.

4.3 The Case of Three Variables

Given is a system of three equations in three variables x, y, z:

$$\begin{aligned} x^2+y^2+z^2 &= A\,, && \text{(stated in step A)} \\ y &= \tfrac{s}{t}x+b\,, && \text{(stated in step B}_1\text{)} \\ z &= \tfrac{s'}{t'}y+c\,, && \text{(stated in step B}_2\text{)} \end{aligned} \tag{53}$$

where s, t, s', and t' are positive integers and b and c are arbitrary numbers (Fig. 12). The numbers $t't$, $t's$ and $s's$ (which are just p_1, p_2 and p_3 in the general setting of section 4.1) are "written down" in step C, in this order, whereas b and c are "written down" in steps C_1 and C_2, respectively.

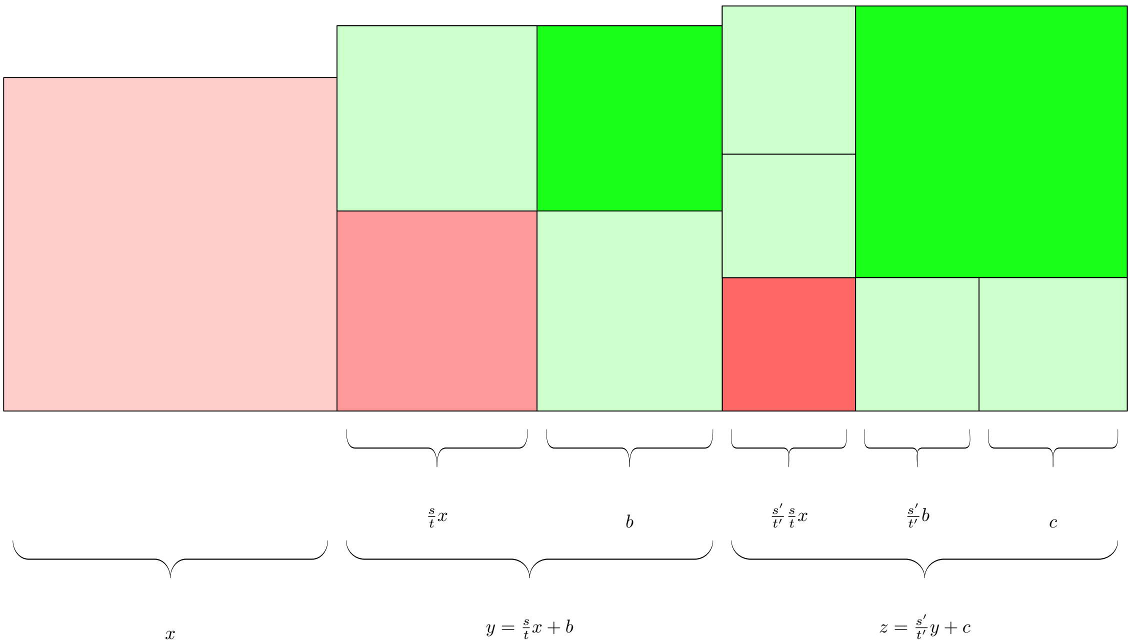


Figure 12

The first step is to to cut all those pieces out of the total figure that are entirely known to us in the sense that they do not contain any variable and so their area can be computed entirely from the givens b and c. These are the two intense green squares in Fig. 12. Their side lengths are b and $\frac{s'}{t'}b+c$ ($\frac{s'}{t'}b$ is computed in step D and $\frac{s'}{t'}b+c$ in step E), and their areas are b^2 (computed in step G) and $\left(\frac{s'}{t'}b+c\right)^2$ (computed in step F). So the total area to be subtracted from the original figure is $b^2+\left(\frac{s'}{t'}b+c\right)^2$ (computed in step H). The resulting figure is shown in Fig. 13.

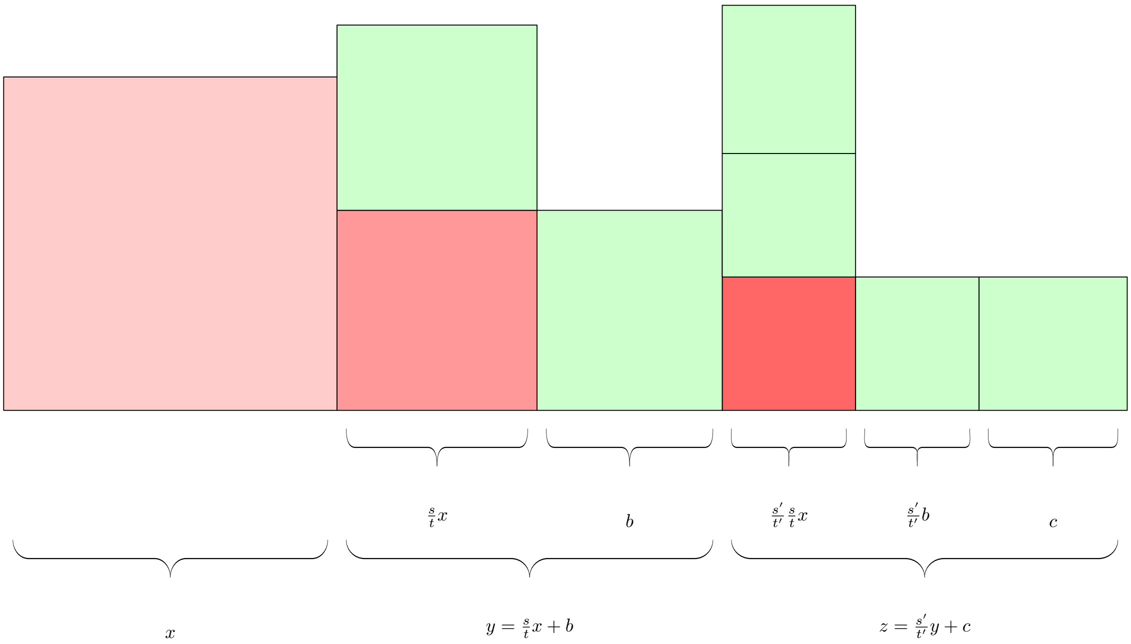


Figure 13

The area of the truncated figure is

$$A' := A - b^2 - \left(\frac{s'}{t'}b + c\right)^2 . \qquad \text{(step I)}$$

Then the left red square of the remaining shape is subdivided into $(t't)^2$ (computed in step J_1) little squares of side lenght $\frac{x}{t't}$, the middle red square into $(t's)^2$ (computed in step J_2), and the right red square into $(s's)^2$ (computed in step J_3) such little squares. Also, each of the two light-green rectangles with length b is split into $t's$ stripes of width $\frac{x}{t't}$ and length b, each of the two light-green rectangles with length $\frac{s'}{t'}b$ is split into $s's$ stripes of width $\frac{x}{t't}$ and length $\frac{s'}{t'}b$, and each of the two light-green rectangles with length c is split into $s's$ stripes of width $\frac{x}{t't}$ and length c (Fig. 14).

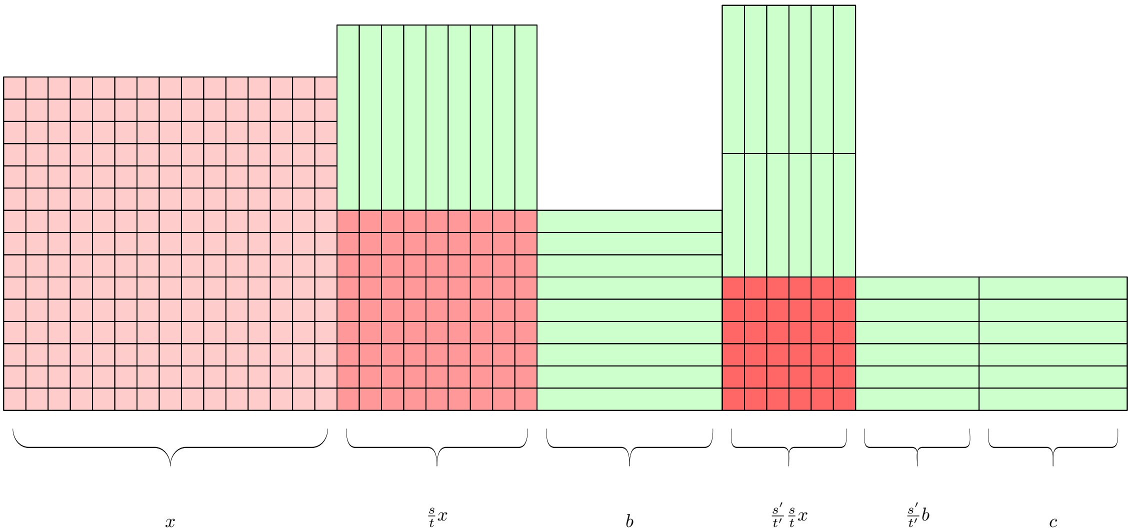


Figure 14

So the truncated figure consists of altogether $[(t't)^2 + (t's)^2 + (s's)^2]$ (computed in step K) little squares with side lenght $\frac{x}{t't}$, plus $t's$ stripes of length b, $s's$ stripes length $\frac{s'}{t'}b$, and $s's$ stripes of length c all of which have width $\frac{x}{t't}$ (Fig. 15).

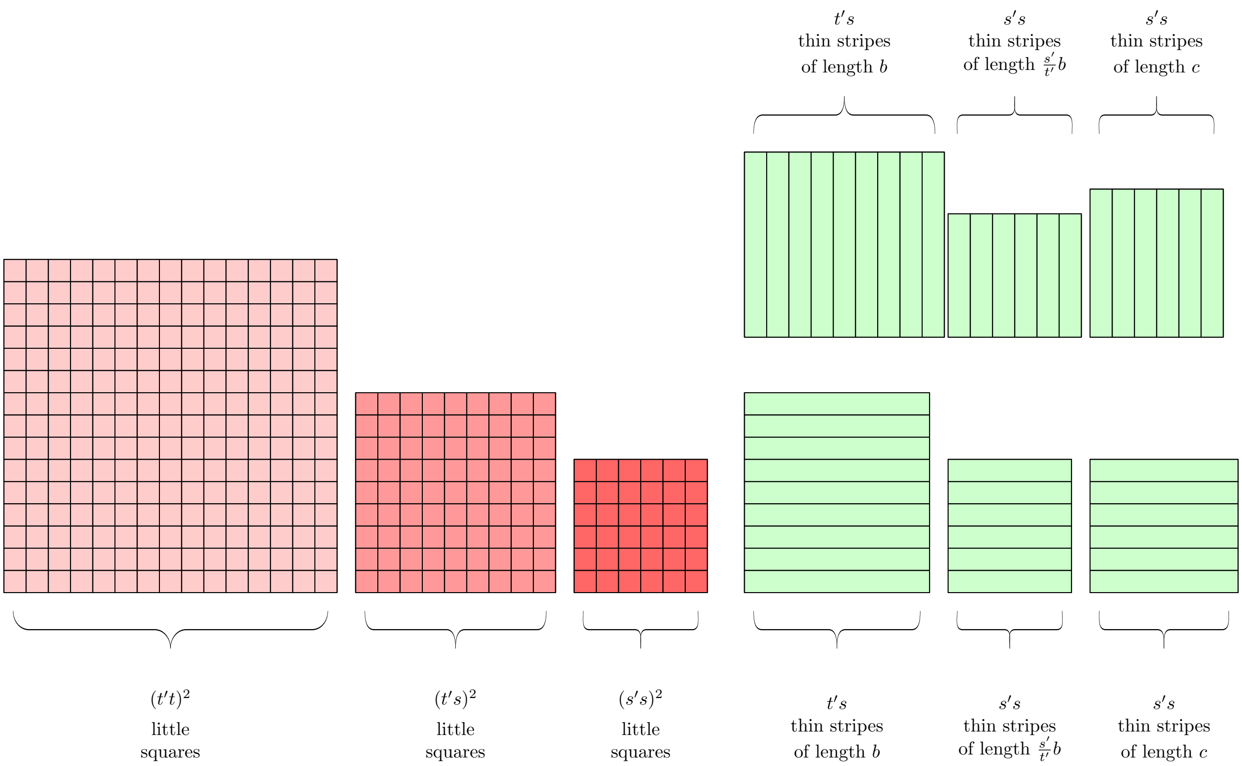


Figure 15

Next imagine all the horizontal stripes arranged into one long super-stripe with width $\frac{x}{t't}$ and length

$$L := (t's)b + (s's)\left(\frac{s'}{t'}b + c\right)$$

and all the vertical stripes into one long super-stripe of the same dimensions. $(t's)b$ is computed in step M_1, $(s's)\left(\frac{s'}{t'}b + c\right)$ is computed in step M_2, and finally L is computed in step N. So far we have only changed the shape of the figure, but not its area. The area of the new arrangement is still A'.

Now the whole arrangement is taken $[(t't)^2 + (t's)^2 + (s's)^2]$ (which is the total number of little squares) times. The resulting entity has now the total area

$$A'' := [(t't)^2 + (t's)^2 + (s's)^2]A' \qquad \text{(step L)}$$

and consists of $\left[(t't)^2 + (t's)^2 + (s's)^2\right]^2$ little squares and $[(t't)^2 + (t's)^2 + (s's)^2]$ horizontal and $[(t't)^2+(t's)^2+(s's)^2]$ vertical super-stripes. They are arranged in the following way: The $\left[(t't)^2 + (t's)^2 + (s's)^2\right]^2$ little squares are made to constitue a super-square, each side of which consists of $[(t't)^2 + (t's)^2 + (s's)^2]$ little squares (of side lenght $\frac{x}{t't}$). So the super-square has a side lengths of $[(t't)^2+(t's)^2+(s's)^2]\frac{x}{t't}$. Now, to each of the $[(t't)^2+(t's)^2+(s's)^2]$ little squares on the right edge of the super-square is attached one of the (altogether $[(t't)^2+(t's)^2+(s's)^2]$) horizontal super-stripes, and to each of the $[(t't)^2 + (t's)^2 + (s's)^2]$ little squares on the top edge of the super-square is attached one of the (altogether $[(t't)^2 + (t's)^2 + (s's)^2]$) vertical super-stripes.

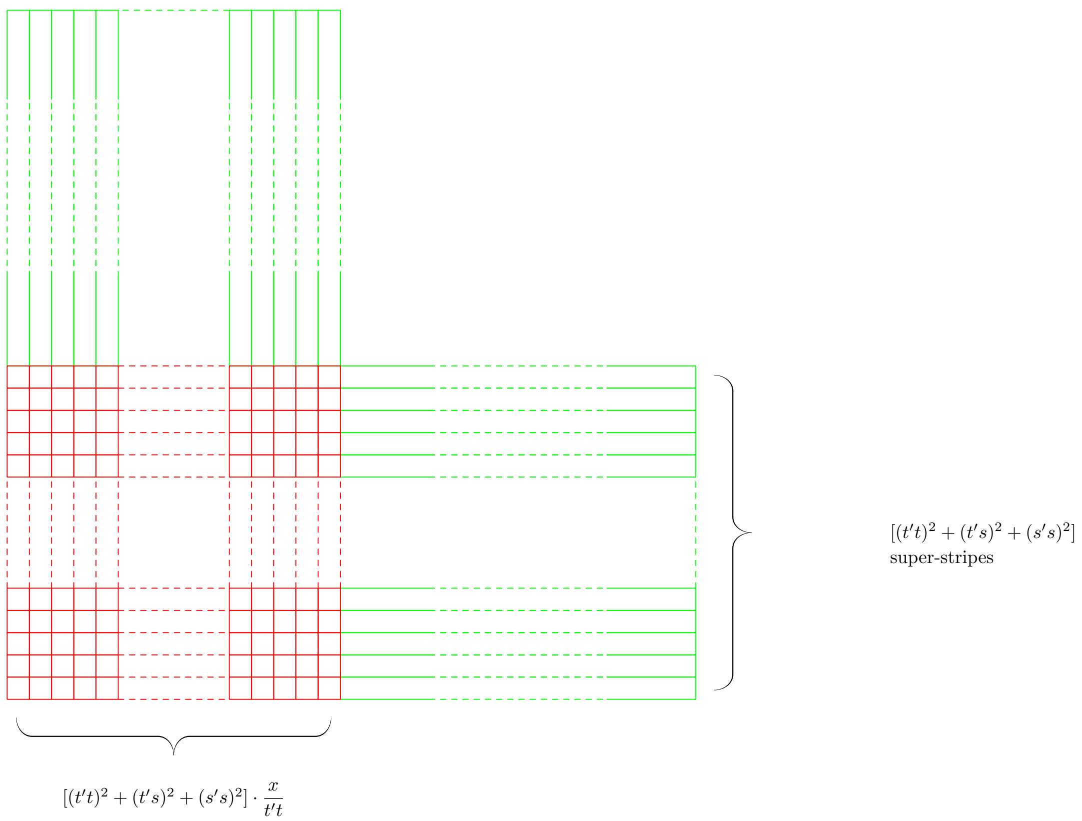


Figure 16

The resulting figure fails to be a square itself by means of a square of side length L and therefore area L^2 which is computed in step O. Adding this missing square we obtain a complete square with total area $A'' + L^2$ (computed in step P) and side length $[(t't)^2 + (t's)^2 + (s's)^2]\frac{x}{t't} + L$. But the side length of a square is just the square root of its area, so we have $[(t't)^2 + (t's)^2 + (s's)^2]\frac{x}{t't} + L = \sqrt{A'' + L^2}$ (computed in step Q), and therefore

$$x = \frac{t't}{(t't)^2 + (t's)^2 + (s's)^2}\left(\sqrt{A'' + L^2} - L\right) .$$

$\left(\sqrt{A'' + L^2} - L\right)$ is computed in step R, $\frac{1}{(t't)^2+(t's)^2+(s's)^2}\left(\sqrt{A'' + L^2} - L\right)$ is computed in step S, and finally x is computed in step T_1.

The solutions for the other two variables are computed by means of $y = \frac{s}{t}x + b$ (computed in step T_2) and $z = \frac{s'}{t'}y + c$ (computed in step T_3).

Remark 4.6 Note that in the following example 01 00 and 40 and 20 are used instead of the minimal possible values for p_1 and p_2 and p_3 which are 6 and 4 and 2. See Remark 1.3 for that. It is also possible, however, that these numbers are to be interpreted as 01 and 00 ▲ 40 and 00 ▲ 20, and that the solution procedure followed a somewhat more direct approach as it was considered in Remark 4.5. This time, this would involve a multiplication by $[1^2 + (\frac{s}{t})^2 + (\frac{s'}{t'})^2]$ in the process. And again, this would not be possible if $[1^2 + (\frac{s}{t})^2 + (\frac{s'}{t'})^2]$ happened to be sexagesimally non-regular and therefore couldn't be written as a sexagesimal fraction. This is, for example, the case in YBC 4714, nos. 4, 7, and 11 (alas without solution procedure), see Neugebauer (1935, 499):

- YBC 4714, no. 4 ($n = 3$): $x_2 = \frac{1}{7}x_1 + 15$, $x_3 = \frac{1}{2}x_2 + 5$,
- YBC 4714, no. 7 ($n = 3$): $x_2 = \frac{1}{11}x_1 + 30$, $x_3 = \frac{1}{7}x_2 + 15$,
- YBC 4714, no. 10 ($n = 4$): $x_2 = x_1 - 10$, $x_3 = \frac{1}{7}x_2 + 25$, $x_4 = \frac{2}{3}x_3$.

4.3.1 Example: BM 13901, no. 24

$$\begin{aligned} x^2 + y^2 + z^2 &= 29\ 10 \\ y &= \tfrac{2}{3}x + 05 \\ z &= \tfrac{1}{2}y + 02\ 30 \end{aligned}$$

Transliteration of BM 13901, no. 24 (rev. ii 17-33) after Neugebauer (1937, 5).

17) **a-ša₃** *ša-la-aš mi-it-h*[*a-r*]*a-ti-i*[*a*] *ak-mur-ma* 29 10
18) *mi-it-har-tum ši-ni-pa-a-at* [*mi-i*]*t-har-tim u₃* 05 **g̃ar**
19) [*mi-š*]*i-i*[*l₅ m*]*i-it-har-tim u₃* 02 [30] **g̃ar** 01 *u₃* ⌜40⌝*u₃* 20
20) 05 *e-le-n*[*u* 40 *t*]*a-la-pa-at* 02 30 *e-le-nu* 20 *ta-la-pa-at*
21) *ba-ma-at* 05 *t*[*e-he*]*-pi* 02 30 *a-na* 02 30 *tu-ṣa-ab* 05 *u₃* 05
22) *tu-uš-ta-kal* 2[5 *t*]*a-la-pa-at* 05 *u₃* 05 *tu-uš-ta-kal*
23) 25 *a-na* 25 *tu-ṣa-ab-ma* 25 25 *lib₃-ba* 29 10 *ta-na-sa₃-ah*

24) 03 45 *ta-la-pa-at* 01 u_3 01 *tu-uš-ta-kal* 01 40 u_3 40 *tu-uš-ta-kal*
25) 26 40 20 u_3 20 *tu-uš-ta-kal* 06 40 u_3 26 40 u_3 01
26) *ta-ka-mar-ma* 01 33 20 *a-*[*na* 03] 45 *ta-na-ši-ma* 05 50
27) 40 *a-na* 05 *t*[*a-n*]*a-ši* 03 20 [20] *a-na* 02 30 *ta-na-ši* 50
28) 03 20 u_3 50 *ta-ka-mar-m*[*a*] 04 10 u_3 04 10 *tu-uš-ta-kal*
29) 17 21 40 *a-na* 05 50 *tu-*[*ṣa*]*-ab-ma* 06 07 21 40 **-e** 19 10 **ib$_2$-sa$_2$**
30) 04 10 *ša tu-uš-ta-ki-lu lib$_3$*[*-bi*] 19 10 *ta-na-sa$_3$-ah-ma* 15 *a-na ši-na e-ṣi-ip* 30
31) *a-na* 01 *ta-na-ši-ma* 30 *mi-i*[*t-h*]*ar-tum iš-ti-a-at* 30 *a-na* 40 *ta-na-ši-ma* 20 u_3 05 *tu-ṣa-ab-ma*
32) 25 *mi-it-har-tum ša-ni-tum ba-ma-at* 25 *te-he-pi* 12 30 u_3 02 30 *tu-ṣa-ab-ma*
33) 15 *mi*[*-it-har-tum ša-lu*]*-uš-tum*

Remark 4.7 There are three original mistakes in the text.

1. First, there is an arithmetical mistake in line 23 where the addition of 25 and 25 results in 25 25 instead of 50 (step H). The reason is probably that it is "two different 25s" (representing two different geometric entities) that are added here. Obviously the scribe erroneously considered them to be on different sexagesimal places

2. Secondly, there is a principal error in line 27 where the length of that part of the super-stripe that comes from the third square has been taken as c instead of as $\frac{b}{2} + c$ (step M_2).

3. And thirdly, in line 30, the wrong number 15 (that emerges as a consecutive error from above) is doubled instead of divided by the total number of small squares, possibly in a desperate attempt to come to the correct result for the square sides (step S).

All the other wrong numbers are consecutive errors obtained by correct calculations. In the following translation the necessary corrections are indicated in an obvious way.

(A) I have added the areas of my three squares and 29 10 (is the result).
(B_1) (The second, respectively third) square side is two thirds of (the preceeding) square side plus 05 added,
(B_2) (respectively) one half of (the preceeding) square side plus 02 ▲ 30 added.
(C) (You write down) 01 00 and 40 and 20.
(C_1) You write down (the) 05 (from line B_1) above (the) 40 (from line C).
(C_2) You write down (the) 02 ▲ 30 (from line B_2) above (the) 20 (from line C).
(D) You break off the half (stated in line B_2) of (the) 05 (from line B_1) (and the result is 02 ▲ 30).
(E) You add (the) 02 ▲ 30 (obtained in step D) to (the) 02 ▲ 30 (from line B_2) (and the result is 05).
(F) You multiply (the) 05 (obtained in step E) and (the) 05 (obtained in step E) (and) write down (the resulting) 25.
(G) You multiply (the) 05 (from line B_1) and (the) 05 (from line B_1) (and the result is 25).
(H) You add (the) 25 (obtained in step G) to (the) 25 (obtained in step F) (and the result is ~~25 25~~ 50).
(I) You tear (the) ~~25 25~~ 50 (obtained in step H) out of the 29 10 (from line A) (and) write down (the resulting) ~~03 45~~ 28 20.

(J_1) You multiply (the) 01 00 and (the) 01 00 (written down in line C) (and the result is) 01 00 00.

(J_2) You multiply (the) 40 and (the) 40 (written down in line C) (and the result is) 26 40.

(J_3) You multiply (the) 20 and (the) 20 (written down in line C) (and the result is 06 40).

(K) You add (the) 06 40 (from step J_3) and (the) 26 40 (from step J_2) and (the) 01 00 00 (from step J_1) and (the result is 01 33 20).

(L) You multiply (the) 01 33 20 (from step K) by (the) ~~03 45~~ 28 20 (obtained in step I) and ~~05 50 00 00~~ 44 04 26 40 (is the result).

(M_1) You multiply (the) 40 (written down in line C) by (the) 05 (from line B_1) (and the result is 03 20).

(M_2) You multiply (the) 20 (written down in line C) by (the) ~~02 ▲ 30 (from line B_2)~~ 05 (obtained in line E) (and the result is) ~~50~~ 01 40.

(N) You add (the) 03 20 (from step M_1) and the ~~50~~ 01 40 (from step M_2) and (the result is ~~04 10~~ 05 00).

(O) You multiply (the) ~~04 10~~ 05 00 (from step N) and (the) ~~04 10~~ 05 00 (from step N) (and the result is ~~17 21 40~~ 25 00 00).

(P) You add (the) ~~17 21 40~~ 25 00 00 (from step O) to (the) ~~05 50 00 00~~ 44 04 26 40 (from line L) and (~~06 07 21 40~~ 44 29 26 40 is the result).

(Q) The square root of ~~06 07 21 40~~ 44 29 26 40 (from step P) is ~~19 10~~ 51 40.

(R) You tear (the) ~~04 10~~ 05 00 that you have multiplied (with itself in step O) out of the (the) ~~19 10~~ 51 40 (from step Q) and (~~15 00~~ 46 40 is the result).

(S) ~~You double (the) 15 00 (obtained in line R)~~ You divide (the) 46 40 (obtained in line R) by 01 33 20 (from step K) (and the result is ~~30 00~~ 00 ▲ 30).

(T_1) You multiply (the) ~~30 00~~ 00 ▲ 30 (from step S) by (the) 01 00 (written down in line C) and (the resulting) 30 is the first square side.

(T_2) You multiply (the) ~~30 00~~ 00 ▲ 30 (from step S) by (the) 40 (written down in line C), and you add (the resulting) 20 and (the) 05 (from line B_1), and (the resulting) 25 is the second square side.

(T_3) You break off the half (from line B_2) of the 25 (from step T_2); you add (the resulting) 12 ▲ 30 and (the) 02 ▲ 30 (from line B_2), and (the resulting) 15 is the third square side.

Remark 4.8 Of course, step (S) should really not read "You divide (the) 46 40 by 01 33 20". Instead, it would be "The inverse of 01 33 20 cannot be solved. What should I multiply by 01 33 20 that gives 46 40?" (compare the examples above).

5 The Type IIc

5.1 The Setting

The general set of equations is now

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \gamma_i x_i = A \tag{54}$$

$$x_i = \frac{s_i}{t_i} x_{i-1} + b_i \qquad \text{for} \quad 2 \leq i \leq n \tag{55}$$

where $s_i, t_i \in \mathbb{Z}, s_i \geq 0, t_i > 0,\ b_i \in \mathbb{R}$ and $A > 0$ are given constants and the x_i are the unknowns asked for. (Without loss of generality, for every i, s_i and t_i can be assumed to have no common divisors, i.e. the fraction $\frac{s_i}{t_i}$ cannot be simplified.)

The linear equations are the same as in the case of Type Ic (section 4.1, there equation 32). Therefore, using the notation introduced in formula (33) in section 4.1, we have again

$$x_i = x_i' + \Delta_i \qquad (1 \leq i \leq n) \tag{56}$$

with

$$x_1' := x_1 \tag{57}$$

$$x_i' := \left(\prod_{j=2}^{i} \frac{s_j}{t_j} \right) x_1 \qquad \text{for} \quad 2 \leq i \leq n \tag{58}$$

and

$$\Delta_1 := 0 \tag{59}$$

$$\Delta_2 := b_2 \tag{60}$$

$$\Delta_i := \frac{s_i}{t_i} \Delta_{i-1} + b_i \qquad \text{for} \quad 3 \leq i < n \tag{61}$$

whence

$$\Delta_i = \sum_{j=3}^{i} \left(\prod_{k=j}^{i} \frac{s_k}{t_k} \right) b_{j-1} + b_i \qquad \text{for} \quad 3 \leq i \leq n\,. \tag{62}$$

So each variable x_i gives rise to (provided the involved respective lengths are non-zero)

- a square with side length x_i which itself is made up from
  - a square of side length x_i' (called "the i-th *base square*" as in section 4.1, red in Fig. 18),

- and two rectangles with side lengths x_i' and Δ_i (light green in Fig. 18),
- and a square of side length Δ_i (intense green in Fig. 18),

just as in the situation of section 4.1, and additionally

- one rectangle with side lengths x_i and γ_i which itself is made up from
  - a rectangle with side lengths x_i' and γ_i (light yellow in Fig. 18),
  - and a rectangle with side lengths Δ_i and γ_i (intense yellow in Fig. 18).

5.2 Solution Procedure

1. Remove from the assembly (wich has total area A) all the elements that are completely determided by the parameters whose values are known (i.e. the s_i, t_i, the b_i, and the γ_i). These are (Fig. 19)

   (a) the squares with side lengths Δ_i and

   (b) the rectangles with side lengths Δ_i and γ_i.

   The area of the remaining assembly is

$$A' := A - \sum_{i=2}^{n} \Delta_i^2 - \sum_{i=2}^{n} \gamma_i \Delta_i \,.$$

2. Split in half all the rectangles that remain from the linear terms of the quadratic equation, i.e. the rectangles with side lengths x_i' and γ_i (the yellow ones in Fig. 20) and reposition them as shown in Fig. 21. (Note that this is exactly the way the linear terms are dealt with in the Old-Babylonian problem texts concerning quadratic equations in one variable, see section B.)

The next two steps are exactly as in section 4.1:

3. The base squares are subdivided into little squares of side length $u = \left(\prod_{i=2}^{n} \frac{1}{t_i}\right) x_1$ and area $S = \left[\left(\prod_{i=2}^{n} \frac{1}{t_i}\right) x_1\right]^2$. This length u fits into x_i', the side length of the i-th base square, p_i times. Again, the p_i satisfy

$$p_1 = \prod_{j=2}^{n} t_j \tag{63}$$

$$p_i = \frac{s_i}{t_i} p_{i-1} \qquad \text{for} \quad 2 \le i \le n \tag{64}$$

   and therefore

$$p_i = \left(\prod_{j=2}^{i} s_j\right)\left(\prod_{k=i+1}^{n} t_k\right) \qquad \text{for} \quad 1 \le i \le n. \tag{65}$$

The total amount Q of little squares is again

$$Q = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} \left[\left(\prod_{j=2}^{i} s_j\right)\left(\prod_{k=i+1}^{n} t_k\right)\right]^2 . \tag{66}$$

4. Every additional rectangle with side lengths x_i' and Δ_i $(i \ge 2)$ is decomposed into p_i stripes each of which has length Δ_i and width u. Note that $u = \frac{1}{p_i} x_i'$ for all i.

In addition to the situation in section 4.1:

5. Every rectangle with side lengths x_i' and $\frac{\gamma_i}{2}$ is decomposed into p_i stripes each of which has length $\frac{\gamma_i}{2}$ and width u.

The result is shown in Fig. 22. Now the construction of the super-structure:

6. Rearrange the assembly in the following way:
   (a) Arrange all the Q little squares into a (horizontal) row.
   (b) Form a horizontal super-stripe and a vertical super-stripe of width u:
      i. Into the horizontal super-stripe enter
         A. all the horizontal stripes emerging from the rectangles with side lengths x_i' and Δ_i,
         B. all the horizontal stripes emerging from the rectangles with side lengths x_i' and $\frac{\gamma_i}{2}$.
      ii. Into the vertical super-stripe enter
         A. all the vertical stripes emerging from the rectangles with side lengths x_i' and Δ_i,
         B. all the vertical stripes emerging from the rectangles with side lengths x_i' and $\frac{\gamma_i}{2}$.

   As already mentioned, the width of each of the two super-stripes is u. The length L of each super-stripe is

$$L = \sum_{i=2}^{n} p_i \Delta_i + \sum_{i=1}^{n} p_i \frac{\gamma_i}{2} .$$

7. Take all this Q times. Arrange the resulting Q^2 little squares into a huge square with side length Qu. Arrange the resulting Q horizontal and the resulting Q vertical super-stripes on the right and atop the huge square respectively (Fig. 23). The new figure has the area

$$A'' := QA'$$

   and the shape of a very huge square of side length $Qu + L$, with a square of side length L missing in the upper right corner.

8. Adding this missing square (which of course has the area L^2) one obtains the complete very huge square (Fig. 6). Its side length is $Qu + L$ and its area is $A''' := A'' + L^2$.

9. Extracting the square root of A''', the new square's area, therefore gives $Qu + L$, its side length:
$$Qu + L = \sqrt{A'''}\,.$$

10. From this square root subtract L, the length of the super-stripe. This gives Qu:
$$Qu = \sqrt{A'''} - L$$
whence $u = \frac{\sqrt{A'''}-L}{Q}$. Since $u = \left(\prod_{i=2}^{n} \frac{1}{t_i}\right) x_1$, the first side length is $x_1 = u \prod_{i=2}^{n} t_i$.
Again, this finally results in
$$x_1 = \frac{\sqrt{QA' + L^2} - L}{Q} \prod_{i=2}^{n} t_i\,.$$
The remaining x_i are computed by means of (55).

The result looks formally like the one in section 4.1 (Type Ic), but note that the expressions for A' and L are now different from those in section 4.1.

Remark 5.1 The method described here also makes sense for the case $n = 1$ (i.e. one variable) where it reduces to the equation $x^2 + \gamma x = A$. Cf remark 6.2 and the examples in appendix B.

Remark 5.2 Remarks 1.2, 4.2 and 4.4 apply accordingly.

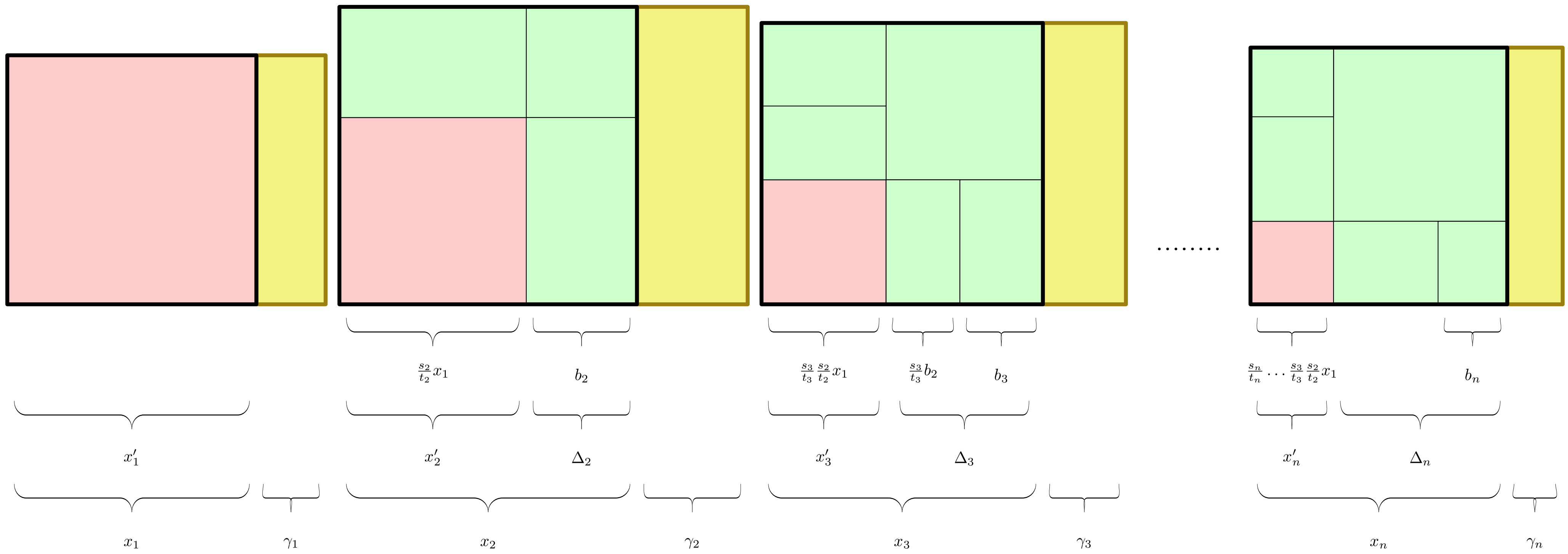


Figure 17: Type IIc: The setting

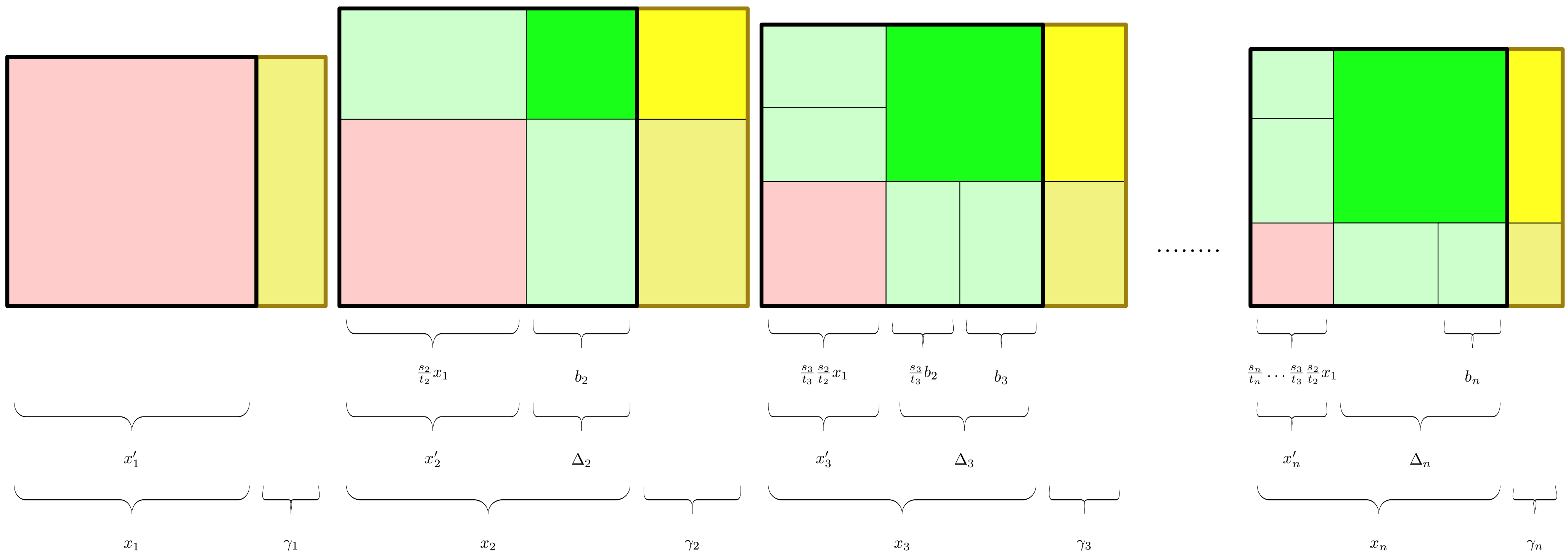


Figure 18: Type IIc: Identifying the constants

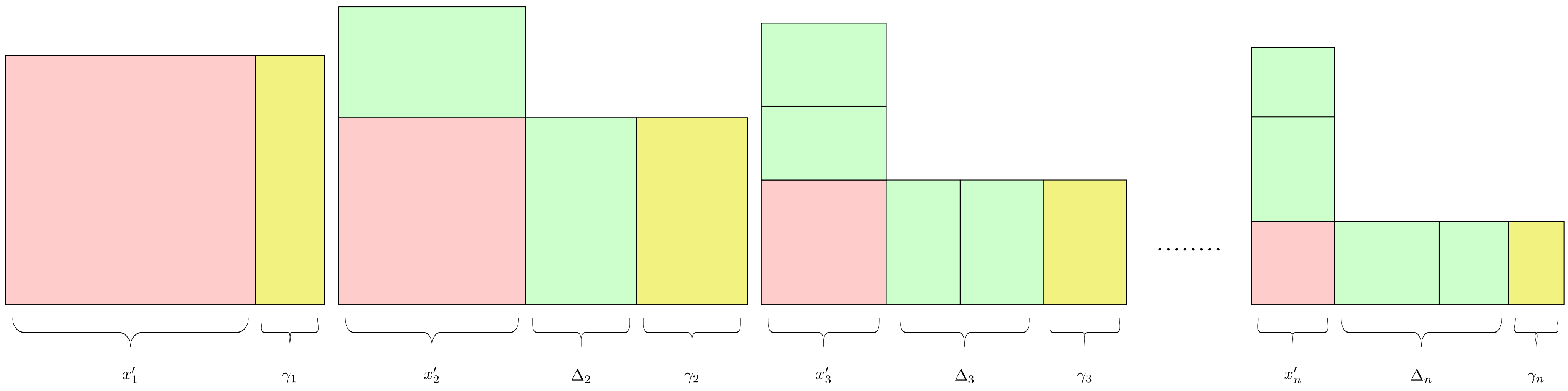


Figure 19: Type IIc: Removing the constants

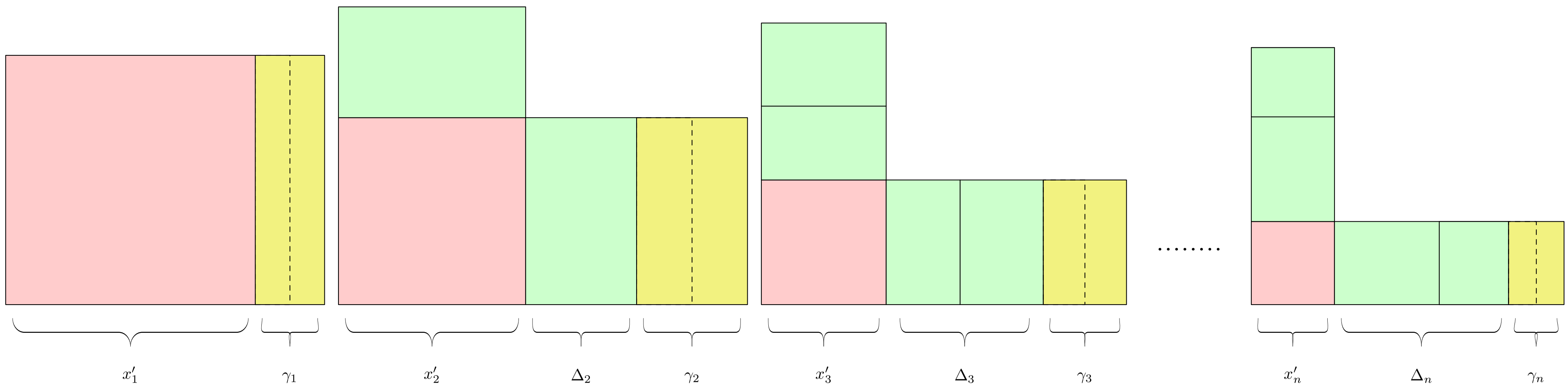


Figure 20: Type IIc: Splitting the linear terms

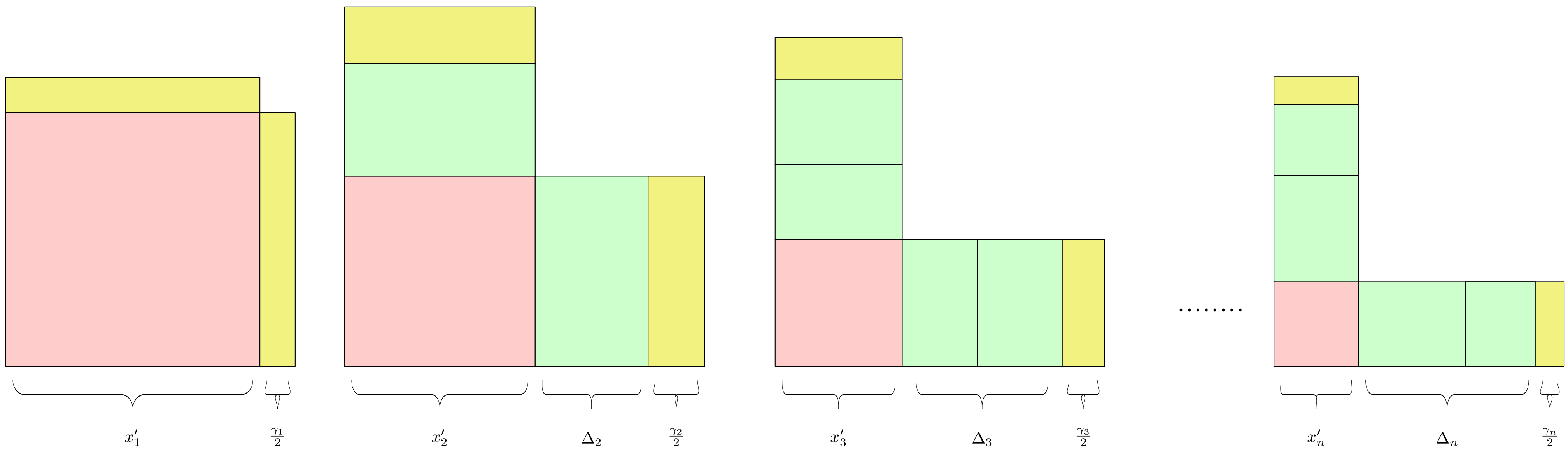


Figure 21: Type IIc: Repositioning the linear terms

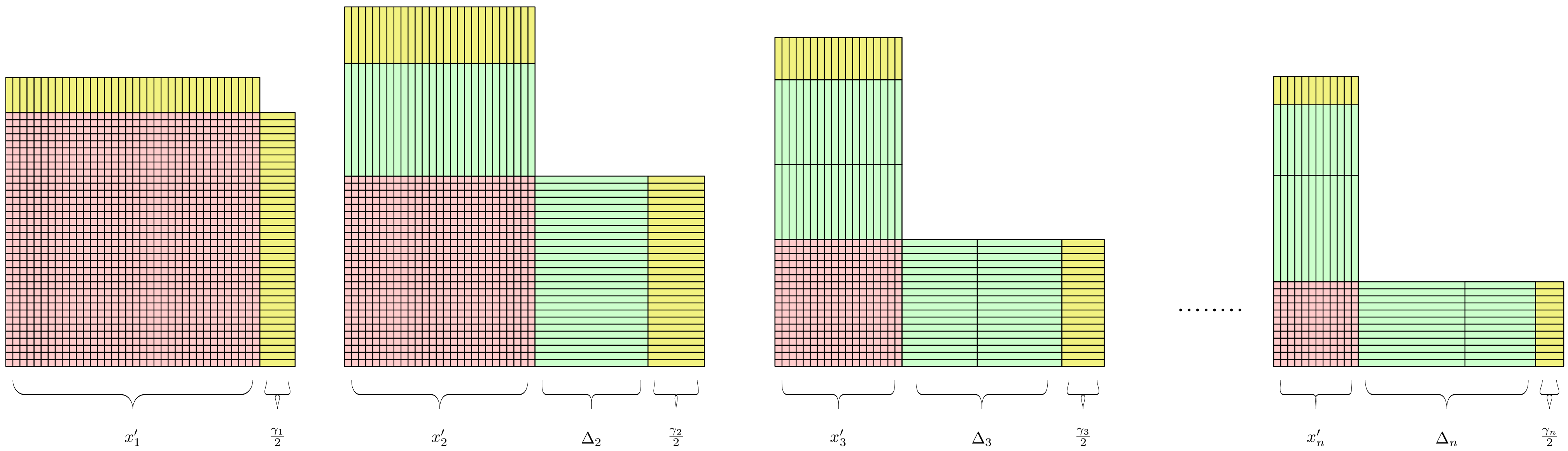


Figure 22: Type IIc: Subdividing

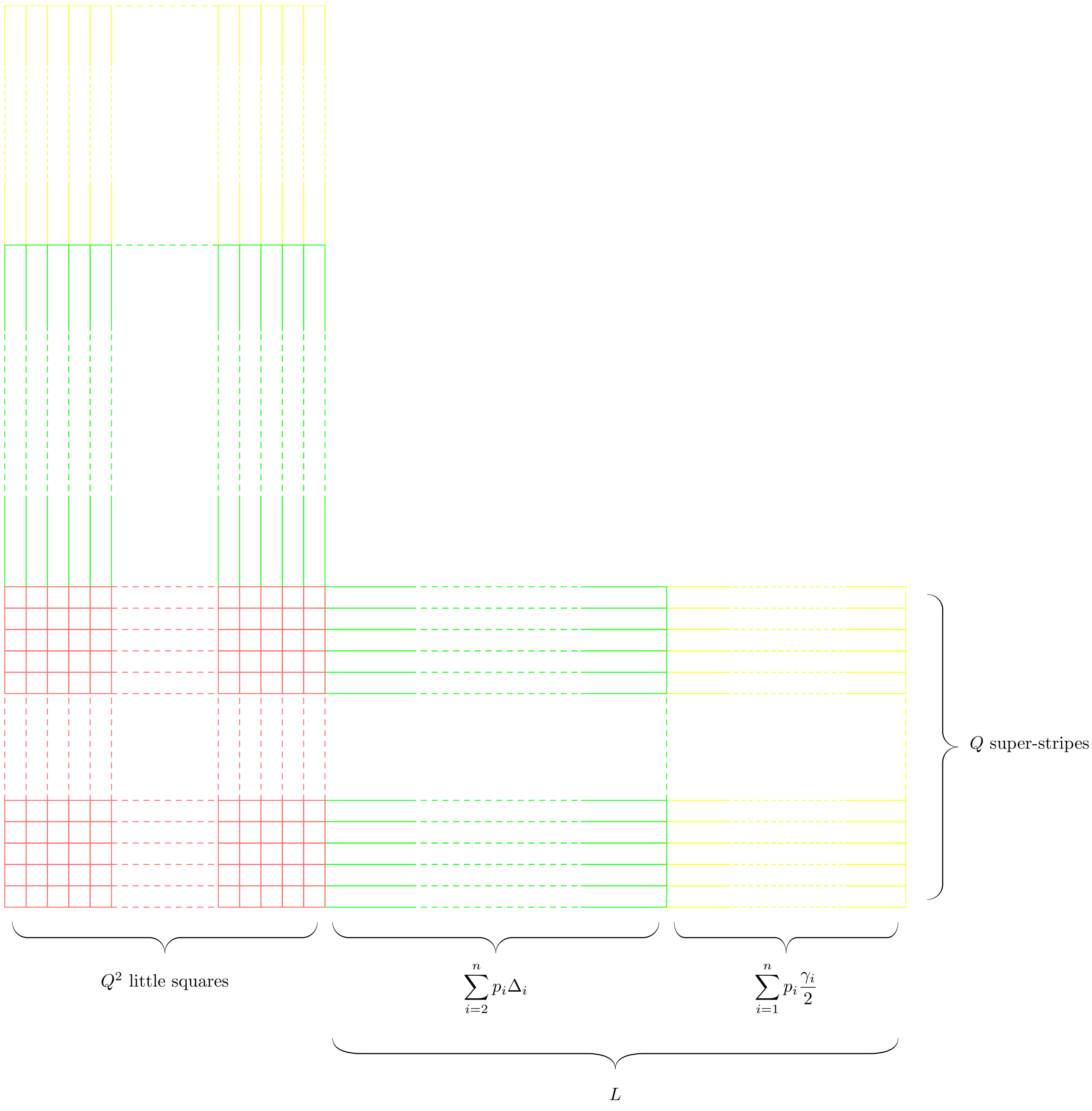


Figure 23: Type IIc: The super-construction

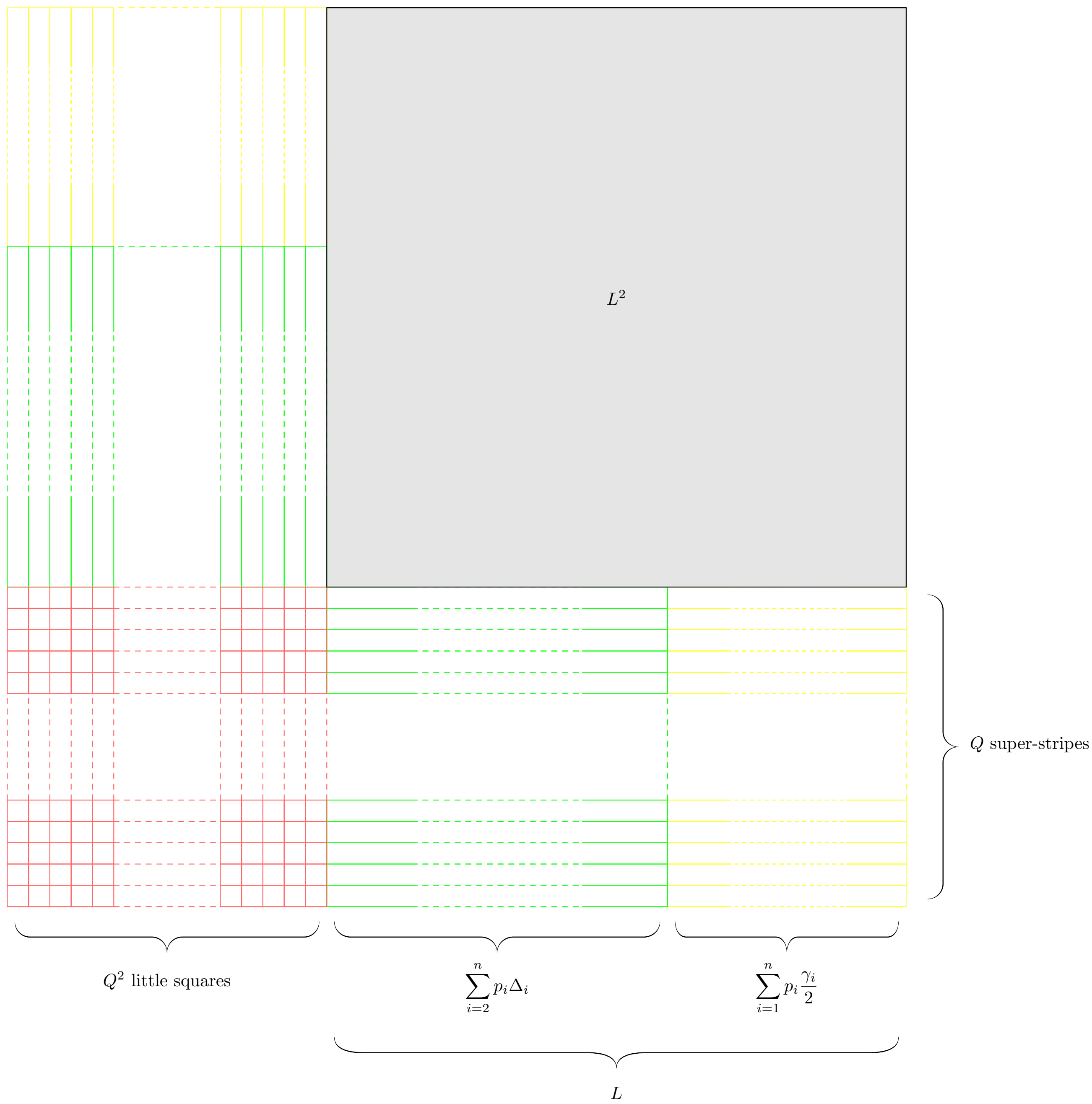


Figure 24: Type IIc: Completing the square

6 The Type IIId

6.1 The Setting

The general set of equations is

$$\sum_{i=1}^{n}\alpha_i x_i^2 + \sum_{1\le i<j\le n}\beta_{ij}x_ix_j + \sum_{i=1}^{n}\gamma_i x_i = A \tag{67}$$

$$x_2 = \frac{s_{21}}{t_{21}}x_1 + b_2 \tag{68}$$
$$x_3 = \frac{s_{31}}{t_{31}}x_1 + \frac{s_{32}}{t_{32}}x_2 + b_3 \tag{69}$$
$$x_4 = \frac{s_{41}}{t_{41}}x_1 + \frac{s_{42}}{t_{42}}x_2 + \frac{s_{43}}{t_{43}}x_3 + b_4 \tag{70}$$
$$\vdots$$
$$x_n = \frac{s_{n1}}{t_{n1}}x_1 + \frac{s_{n2}}{t_{n2}}x_2 + \frac{s_{n3}}{t_{n3}}x_3 + \ldots + \frac{s_{n,n-1}}{t_{n,n-1}}x_{n-1} + b_n \tag{71}$$

where $\alpha_i, \beta_{ij}, s_{ij}, t_{ij} \in \mathbb{Z}$ (with $s_{ij} \ge 0, t_{ij} > 0$), $b_i, \gamma_i \in \mathbb{R}$ and $A > 0$ are given constants and the x_i are the unknowns asked for.[7] (Again, without loss of generality, for every i, j, s_{ij} and t_{ij} can be assumed to have no common divisors, i.e. the fraction $\frac{s_{ij}}{t_{ij}}$ cannot be simplified.) Even though there are no principal restrictions on the range of the parameters, one must of course make sure that the resulting system is solvable. These equations lead to

$$\begin{aligned}
x_2 &= \frac{s_{21}}{t_{21}}x_1 + b_2\\
x_3 &= \frac{s_{31}t_{32}t_{21} + t_{31}s_{32}s_{21}}{t_{31}t_{32}t_{21}}x_1 + \frac{s_{32}}{t_{32}}b_2 + b_3\\
x_4 &= \frac{s_{41}t_{42}t_{43}t_{31}t_{32}t_{21} + t_{41}t_{42}s_{43}s_{31}t_{32}t_{21} + t_{41}t_{42}s_{43}t_{31}s_{32}s_{21} + t_{41}s_{42}t_{43}t_{31}t_{32}s_{21}}{t_{41}t_{42}t_{43}t_{31}t_{32}t_{21}}x_1\\
&\quad + \frac{s_{42}t_{43}t_{32} + t_{42}s_{43}s_{32}}{t_{42}t_{43}t_{32}}b_2 + \frac{s_{43}}{t_{43}}b_3 + b_4\\
&\vdots
\end{aligned} \tag{72}$$

In order to bring this into a compact formal notation a few definitions are made. First, let

$$\begin{aligned}
T &:= (t_{n1}t_{n2}\ldots t_{n,n-1})(t_{n-1,1}t_{n-1,2}\ldots t_{n-1,n-2})\ldots(t_{31}t_{32})(t_{21})\\
&= \prod_{i=2}^{n}\prod_{j=1}^{i-1} t_{ij}.
\end{aligned} \tag{73}$$

[7] The expression $\sum_{1\le i<j\le n}\beta_{ij}x_ix_j$ is short for $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\beta_{ij}x_ix_j$. It is sufficient to consider x_ix_j for $i < j$ because $x_jx_i = x_ix_j$.

Then, for any two natural numbers i and j with $i > j$ let $\mathcal{J}_{i|j}$ denote the set of all strictly decreasing (finite) sequences $(i, \sigma_1, \ldots, \sigma_k, j)$ (i.e. $(i > \sigma_1 > \ldots > \sigma_k > j)$) of natural numbers.

Examples: $\mathcal{J}_{4|3} = \{(4,3)\}$, $\mathcal{J}_{5|2} = \{(5,4,3,2),(5,4,2),(5,3,2),(5,2)\}$, and $\mathcal{J}_{8|4} = \{(8,7,6,5,4),(8,7,6,4),(8,7,5,4),(8,6,5,4),(8,7,4),(8,6,4),(8,5,4),(8,4)\}$.

For any given $\tau = (i, \sigma_1, \ldots, \sigma_k, j) \in \mathcal{J}_{i|j}$ let

$$\tau(T) := T \cdot \frac{s_{i\sigma_1} s_{\sigma_1\sigma_2} s_{\sigma_2\sigma_3} \ldots s_{\sigma_{k-1}\sigma_k} s_{\sigma_k j}}{t_{i\sigma_1} t_{\sigma_1\sigma_2} t_{\sigma_2\sigma_3} \ldots t_{\sigma_{k-1}\sigma_k} t_{\sigma_k j}}\,, \tag{74}$$

i.e. the result of replacing, in T, each of the $s_{i\sigma_1}, s_{\sigma_1\sigma_2}, s_{\sigma_2\sigma_3}, \ldots s_{\sigma_{k-1}\sigma_k}, s_{\sigma_k j}$ by the corresponding $t_{i\sigma_1}, t_{\sigma_1\sigma_2}, t_{\sigma_2\sigma_3} \ldots t_{\sigma_{k-1}\sigma_k}, t_{\sigma_k j}$.

Example: Let $\tau = (8,7,5,4) \in \mathcal{J}_{8|4}$. Then $\tau(T) = T \cdot \dfrac{s_{87}s_{75}s_{54}}{t_{87}t_{75}t_{54}}$.

Finally, for any two natural numbers i and j with $i > j$ let

$$S_{i|j} \;:=\; \sum_{\tau \in \mathcal{J}_{i|j}} \tau(T)\,. \tag{75}$$

With these definitions made, the relations in formula (72) read

$$\begin{aligned}
x_2 &= \frac{S_{2|1}}{T}x_1 + \frac{T}{T}b_2 \\
x_3 &= \frac{S_{3|1}}{T}x_1 + \frac{S_{3|2}}{T}b_2 + \frac{T}{T}b_3 \\
x_4 &= \frac{S_{4|1}}{T}x_1 + \frac{S_{4|2}}{T}b_2 + \frac{S_{4|3}}{T}b_3 + \frac{T}{T}b_4 \\
&\vdots \\
x_k &= \frac{S_{k|1}}{T}x_1 + \frac{S_{k|2}}{T}b_2 + \frac{S_{k|3}}{T}b_3 + \ldots + \frac{S_{k|k-1}}{T}b_{k-1} + \frac{T}{T}b_k \\
&\vdots \\
x_n &= \frac{S_{n|1}}{T}x_1 + \frac{S_{n|2}}{T}b_2 + \frac{S_{n|3}}{T}b_3 + \ldots + \frac{S_{n|n-1}}{T}b_{n-1} + \frac{T}{T}b_n
\end{aligned} \tag{76}$$

With the additional convention $S_{i|i} := T$, for all i, we finally obtain

$$x_i \;=\; \frac{1}{T}\left(S_{i|1}x_1 + \sum_{j=2}^{i} S_{i|j}b_j\right) \qquad \text{for all } 1 \le i \le n\,. \tag{77}$$

Using the notation introduced in formula (33) in section 4.1, this reads as

$$x_i \;=\; x_i' + \Delta_i \tag{78}$$

where now

$$x'_i \;=\; \frac{S_{i|1}}{T} x_1 \qquad \text{for all } 1 \le i \le n \tag{79}$$

(especially $x'_1 = x_1$) and

$$\Delta_i \;=\; \frac{1}{T} \sum_{j=2}^{i} S_{i|j} b_j \qquad \text{for all } 1 \le i \le n\,. \tag{80}$$

Note that from this we automatically recover $\Delta_1 = 0$ and $\Delta_2 = b_2$ (cf formulae (36) and (37) in section 4.1).

So each variable x_i gives rise to (provided the involved respective lengths are non-zero)

- α_i copies of a square with side length x_i which itself is made up from
  - a square of side length x'_i (called "the i-th *base square*" as in section 4.1),
  - and two rectangles with side lengths x'_i and Δ_i,
  - and a square of side length Δ_i,

  just as in the situation of section 4.1;
- one rectangle with side lengths x_i and γ_i which itself is made up from
  - a rectangle with side lengths x'_i and γ_i,
  - and a rectangle with side lengths Δ_i and γ_i.

Furthermore, every pair of variables x_i and x_j with $i < j$ gives rise to

- β_{ij} copies of a rectangle with side lengths x_i and x_j which itself is made up from
  - a rectangle with side lengths x'_i and x'_j,
  - a rectangle with side lengths x'_i and Δ_j,
  - a rectangle with side lengths x'_j and Δ_i,
  - a rectangle with side lengths Δ_i and Δ_j.

6.2 Solution Procedure

1. Remove from the assembly (wich has total area A) all the elements that are completely determided by the parameters whose values are known (i.e. the $\alpha_i, \beta_{ij}, \gamma_i$ as well as the s_i, t_i and the b_i). These are
   (a) all the squares with side lengths Δ_i (from all the α_i copies of the original unknown squares with side lengths x_i),
   (b) all the rectangles with side lengths Δ_i and γ_i,
   (c) all the rectangles with side lengths Δ_i and Δ_j (from all the β_{ij} copies of the rectangles with side lengths x_i and x_j).

The area of the remaining assembly is

$$A' \;:=\; A - \sum_{i=2}^{n} \alpha_i\, \Delta_i^2 - \sum_{i=1}^{n} \gamma_i \Delta_i - \sum_{i<j} \beta_{ij} \Delta_i \Delta_j \,.$$

2. The base squares are subdivided into little squares of side length $\frac{x_1}{T}$ and area $\left(\frac{x_1}{T}\right)^2$. This length $\frac{x_1}{T}$ fits into x_i', the side length of the i-th base square, $S_{i|1}$ times. Every copy of the i-th base square therefore gives rise to $S_{i|1}^2$ little squares of side length $\frac{x_1}{T}$.

3. Correspondingly, a rectangle with side lengths x_i' and x_j' gives rise to $S_{i|1} S_{j|1}$ such little squares of side length $\frac{x_1}{T}$.

The total amount Q of little squares is therefore

$$Q \;=\; \sum_{i=1}^{n} \alpha_i\, S_{i|1}^2 \;+\; \sum_{i<j} \beta_{ij}\, S_{i|1}\, S_{j|1} \,.$$

4. Every rectangle with side lengths x_i' and Δ_i ($i \geq 2$) is decomposed into $S_{i|1}$ stripes each of which has length Δ_i and width $\frac{x_1}{T}$.

5. Every rectangle with side lengths x_i' and γ_i is decomposed into $S_{i|1}$ stripes each of which has length γ_i and width $\frac{x_1}{T}$.

6. Finally, for all i, j with $i < j$,

   (a) every rectangle with side lengths x_i' and Δ_j is decomposed into $S_{i|1}$ stripes each of which has length Δ_j and width $\frac{x_1}{T}$, and

   (b) every rectangle with side lengths x_j' and Δ_i is decomposed into $S_{j|1}$ stripes each of which has length Δ_i and width $\frac{x_1}{T}$.

7. Rearrange the assembly in the following way:

   (a) Arrange all the Q little squares into a (horizontal) row.

   (b) Form a horizontal super-stripe and a vertical super-stripe:

      i. Into the horizontal super-stripe enter

         A. all the stripes emerging from the horizontal rectangles with side lengths x_i' and Δ_i,

         B. one half of every stripe with length γ_i and width $\frac{x_1}{T}$,

         C. one half of every stripe emerging from the rectangles with side lengths x_i' and Δ_j and from the rectangles with side lengths x_j' and Δ_i.

      ii. Into the vertical super-stripe enter

         A. all the stripes emerging from the vertical rectangles with side lengths x_i' and Δ_i,

         B. the second half of every stripe with length γ_i and width $\frac{x_1}{T}$,

         C. the second half of every stripe emerging from the rectangles with side lengths x_i' and Δ_j and from the rectangles with side lengths x_j' and Δ_i.

The width of each of the two super-stripes is $\frac{x_1}{T}$. The length L of each super-stripe is

$$L = \sum_{i=1}^{n} \left(\alpha_i \, S_{i|1} \, \Delta_i \; + \; S_{i|1} \, \frac{\gamma_i}{2} \right) \; + \; \sum_{i<j} \beta_{ij} \left(S_{i|1} \, \frac{\Delta_j}{2} \; + \; S_{j|1} \, \frac{\Delta_i}{2} \right) .$$

8. Take all this Q times. Arrange the resulting Q^2 little squares into a huge square with side length $Q\frac{x_1}{T}$. Arrange the resulting Q horizontal and the resulting Q vertical super-stripes on the right and atop the huge square respectively. The new figure has the area

$$A'' \; := \; Q A'$$

and the shape of a very huge square of side length $Q\frac{x_1}{T} + L$, with a square of side length L missing in the upper right corner.

9. Adding this missing square (which of course has the area L^2) one obtains the complete very huge square of side length $Q\frac{x_1}{T} + L$. The area of the very huge square is $A''' := A'' + L^2$. Therefore we have $\sqrt{A'''} = Q\frac{x_1}{T} + L$ and from this

$$x_1 \; = \; T \, \frac{\sqrt{A'''} - L}{Q} \, .$$

The remaining x_i are compuded by means of (68)-(71).

Remark 6.1 Remarks 1.2, 4.2 and 4.4 apply accordingly.

6.3 Final Remark: Reduction of Types

In the special case that $\alpha_i = 1$ and $\gamma_i = 0$ for all i, $\beta_{ij} = 0$ for all $i < j$, and $s_{ij} = 0, t_{ij} = 1$ whenever $j \neq i-1$, the problem reduces to the situation in section 4.1. So does the solution procedure as can be seen as follows:

The $s_{i,i-1}$ (now the only non-zero ones among the s_{ij}) and the $t_{i,i-1}$ are precisely the s_i and t_i from section 5. With this notational replacement understood, T (defined in (73)) reduces to

$$T \; = \; \prod_{i=2}^{n} t_i \; = \; t_2 \, t_3 \, \ldots \, t_n$$

and each of the $S_{i|j}$ defined in (75) reduces to the single term

$$S_{i|j} \; = \; T \cdot \frac{s_i \, s_{i-1} \, \ldots \, s_{j+1}}{t_i \, t_{i-1} \, \ldots \, t_{j+1}} \; = \; T \prod_{k=j+1}^{i} \frac{s_k}{t_k} \; = \; t_2 \ldots t_j \, s_{j+1} \ldots s_i \, t_{i+1} \ldots t_n$$

because all the other summands of (75) contain at least one zero factor in the numerator and therefore vanish. So we get from (79):

$$x'_i \; = \; \frac{S_{i|1}}{T} x_1 \; = \; \left(\prod_{j=2}^{i} \frac{s_j}{t_j} \right) x_1$$

for all $2 \leq i \leq n$ as in (35), and from (80):

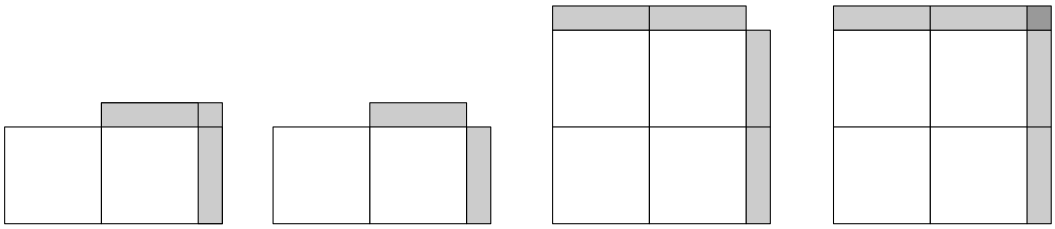
$$\begin{aligned}\Delta_i &= \frac{1}{T}\sum_{j=2}^{i} S_{i|j} b_j &= \sum_{j=2}^{i}\left(\prod_{k=j+1}^{i}\frac{s_k}{t_k}\right) b_j = \sum_{j=3}^{i+1}\left(\prod_{k=j}^{i}\frac{s_k}{t_k}\right) b_{j-1} \\ & &= \sum_{j=3}^{i}\left(\prod_{k=j}^{i}\frac{s_k}{t_k}\right) b_{j-1} + b_i\end{aligned} \tag{81}$$

for all $3 \leq i \leq n$, as in (38). The remaining steps are obvious. Of course, if we do not demand $\gamma_i = 0$ for all i, we obtain the situation and procedure from section 5.

Remark 6.2 The method described here also makes sense for the case $n = 1$ (i.e. one variable) where it reduces to the equation $\alpha x^2 + \gamma x = A$. In this case step 2 gives α "little" squares which are copies of the unknown square with side length x. The super-stripe has therefore width x and length $\frac{\gamma}{2}$. For an example of this case with $\alpha \neq 1$ see appendix B.3.

A Supplement to Type Ib: A Variant in Two Variables

Of course, the method described in section 2 also works for the case of two squares. The corresponding sequence of drawings is[8]



starting point | removing b^2 | taking the area two (the number of squares) times | completing the square

However, BM 13901 no. 9 uses a different method to solve the problem

$$x^2 + y^2 = A \quad (82)$$

$$y = x + b \quad (83)$$

BM 13901, no. 9 (obv. ii 3-10)[9] (after Neugebauer (1937, 2))

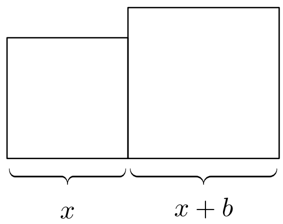
3) **a-ša$_3$** *ši-ta mi-it-ha-ra-ti-ia ak-mur-ma* 21 40
4) *mi-it-har-tum* **ugu** *mi-it-har-tim* 10 *i-te-er*
5) *ba-ma-at* 21 40 *te-he-pi-ma* 10 50 *ta-la-pa-at*
6) *ba-ma-at* 10 *te-he-pi* 05 **u$_3$** 05 *tu-uš-ta-kal*
7) 25 *lib$_3$-bi* 10 50 *ta-na-sa$_3$-ah-ma* 10 25-**e** 25 **ib$_2$-sa$_2$**
8) 25 *a-di ši-ni-šu ta-la-pa-at* 05 *ša tu-uš-ta-ki-lu*
9) *a-na* 25 *iš-te-en tu-ṣa-ab-ma* 30 *mi-it-har-tum*
10) 05 *lib$_3$-bi* 25 *ša-ni-im ta-na-sa$_3$-ah-ma* 20 *mi-it-har-tum ša-ni-tum*

(A) I have added the areas of my two squares and 21 40 (was the result).
(B) (The one) square side exceeds (the other) square side by 10.
(C) You break off the half of the 21 40 (from line A).
(D) You record (the resulting) 10 50.
(E) You break off the half of the 10 (from line B).
(F) You multiply (the resulting) 5 and (the resulting) 5.
(G) You tear (the resulting) 25 out of the 10 50 (recorded in step D).
(H) (Of the resulting) 10 25 the square root is 25.
(I) You record (this) 25 twice.
(J) You add the 5 that you have squared (in step F), to the one 25 (recorded in step I), and
(K) (the resulting) 30 is (the first) square side.

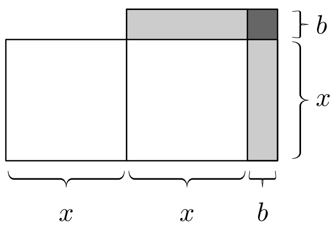
[8] The material for this section is taken from Brunke (2017, 10-12).
[9] See Høyrup (2002, 68-70).

(L) You subtract the 5 (that you have squared in step F) from the second 25 (recorded in step I), and

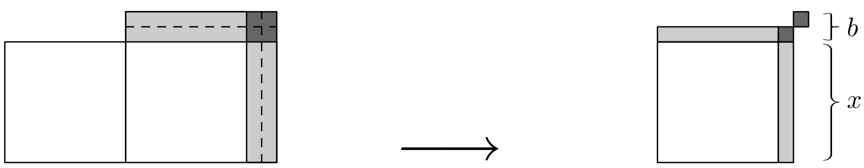
(M) (the resulting) 20 is the second square side.



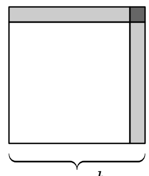
Eqn. (82) is stated in step (A), eqn. (83) in step (B). The total area of the two unknown squares is A (here $A = 21\ 40$) and the second square side exceeds the first one by b (here $b = 10$). The surplus of the second square over the first one consists of two rectangular stripes of thickness b (light grey) and a square with side lenght b in the upper right corner (dark grey):



In step (C) the total area is halved (i.e. one half of it is "broken off"). This can be realized by removing one of the two white squares, one half of each of the two light-grey stripes, and two quaters (= one half) of the dark-grey square:



The new figure (on the right side) has the area $A' := \frac{1}{2}A$. Each of the two little remaining dark-grey squares has the side length $\frac{b}{2}$ which is computed in step (E) and squared in step (F) thus giving the area $(\frac{b}{2})^2$ of each of the two little remaining dark-grey squares. In step (G) one of them is removed ("torn out")



leaving us with a square that has side length $x + \frac{b}{2}$ and area $A'' := A' - (\frac{b}{2})^2$. Therefore, extracting the square root from A'' (step H) gives $x + \frac{b}{2}$. This intermediary result is "recorded twice" (step I) and undergoes two different operations: adding $\frac{b}{2}$ (step J) gives us $x + b$ which is the side lenght of the bigger square (step K); and subtracting $\frac{b}{2}$ (step L) gives us x which is the side lenght of the smaller square (step M).

Remark A.1 Note that there is no mention here of something like "02 (which is the number of the) squares". And indeed, it turns out that this method has no immediate analog for the cases of $3, 4, 5, \ldots$ squares. But it is simpler than the $n = 2$ equivalent of the $n = 3$ problem discussed above, which might be the reason that it has been used instead.

B Dealing with Linear Terms: Quadratic Equations in One Variable

This section is about solving *one* quadratic equation that contains a linear term:

$$x^2 + \gamma x \;=\; A$$

where γx is called the "linear term" of the equation wheras x^2 is the quadratic term. The desire to solve such an equation geometrically (which amounts to interpreting x^2 as a "real" square with side length x) makes it necessary to have a means to geometrically add/subtract a length to/from an area. This means is the "rectanglification" of the length, i.e. the length is turned into (or: replaced by) a rectangular area whose sides are the given length and the unit width 1. The (numerical) area measure of this rectangle is the same as the length measure of the given length.

Given a length ℓ one might call the result of its rectanglification the *unit rectangle generated by (erected over)* ℓ and denote it by $\mathcal{U}(\ell)$. The universal unit width whose purpose is to provide this function of rectanglification is called the *waṣītum*.

B.1 The Case $\gamma > 0$

B.1.1 $\gamma = 1: \quad x^2 + x = A$

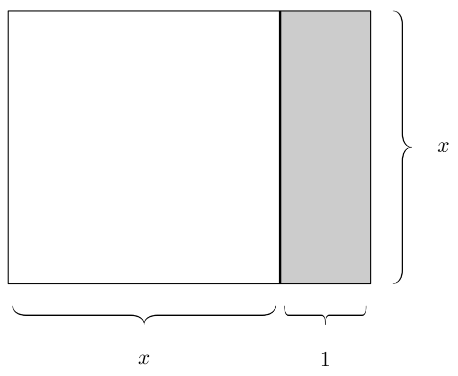
BM 13901, no. 1 (obv. i 1-4)[10] (after Neugebauer (1937, 1))

1) **a-ša₃**$^{[lam]}$ *u₃ mi-it-har-ti ak-m*[*ur-m*]*a* 45-**e** 01 *wa-ṣi-tam*
2) *ta-ša-ka-an ba-ma-at* 01 *te-he-pi* [30] *u₃* 30 *tu-uš-ta-kal*
3) 15 *a-na* 45 *tu-ṣa-ab-ma* 01-**e** 01 **ib₂-si₈** 30 *ša tu-uš-ta-ki-lu*
4) *lib₃-ba* 01 *ta-na-sa₃-ah-ma* 30 *mi-it-har-tum*

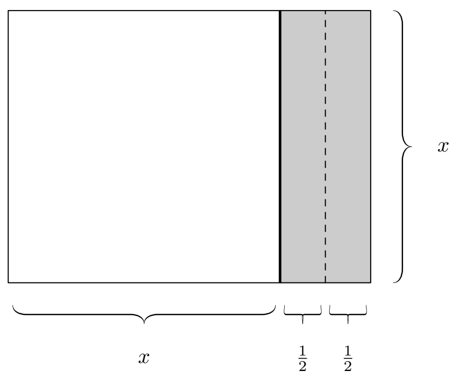
(A) I have added the area (of my square) and my square-side and 00▴45 (was the result).
(B) You put 01, the *waṣītum*.
(C) You break off the half of (this) 01 (i.e. of the *waṣītum*).
(D) You multiply (the resulting) 00▴30 and (the resulting) 00▴30.
(E) You add (the resulting) 00▴15 to the 00▴45 (from step A) and
(F) the square root of (the resulting) 01 is 01.
(G) You tear the 00▴30 that you have multiplied (with itself in step D) out of the 01 (obtained in step F)
(H) and (the resulting) 00▴30 is the square side.

The figure consisting of the square with side length x (and therefore area x^2) and the unit rectangle generated by x (the grey-shaded stripe of width 1) has the total area A:

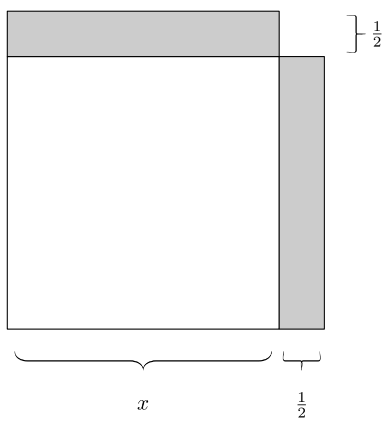
[10] See Høyrup (2002, 50-52).



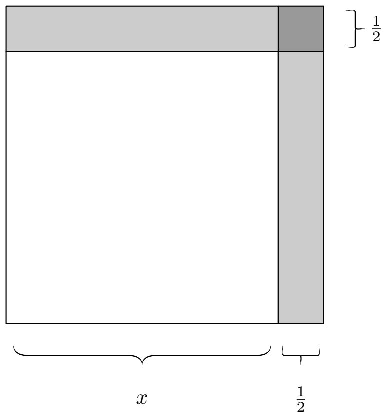
The grey-shaded stripe is imagined to be cut into two halves of width $\frac{1}{2}$ each:



The two halves are then thought to be arranged as follows:



The result is a square with side length $x+\frac{1}{2}$, with a square of side length $\frac{1}{2}$ (and therefore area $(\frac{1}{2})^2$) missing in the upper right corner. This side length $\frac{1}{2}$ is computed in step (C) by cutting the *waṣītum* in half, and squared in step (D) in order to get the area of the square missing in the upper right corner. So far the total area is still A. Then in step (E) this missing square is added and one gets a complete square with side length $x + \frac{1}{2}$ and area $A' := A + (\frac{1}{2})^2$



whence extracting the square root of A' (step F) yields $x + \frac{1}{2}$. Subtracting $\frac{1}{2}$ (step G; note that the wording identifies this $\frac{1}{2}$ as the side length of the square formerly missing in the upper right corner of the figure) finally gives x as stated in line (H).

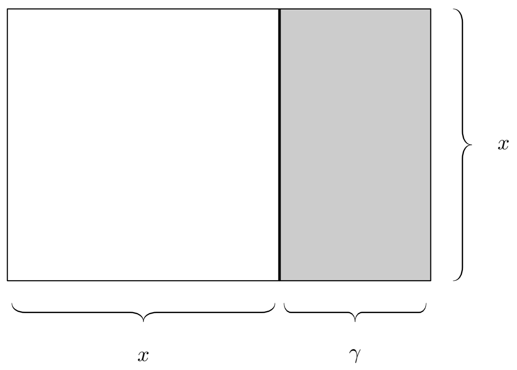
B.1.2 $\gamma \neq 1: \quad x^2 + \frac{2}{3}x = A$

BM 13901, no. 6 (obv. i 30-34) (after Neugebauer (1937, 2))

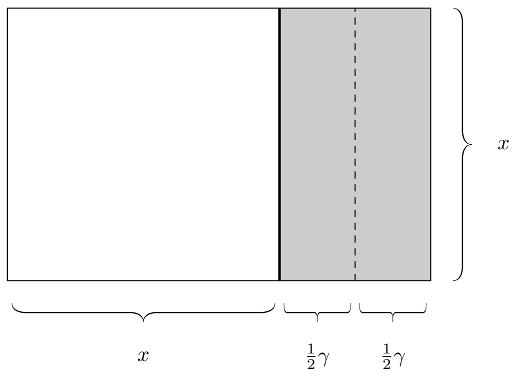
30) [**a-ša$_3$**^{lam} *u$_3$ ši-ni-pa-a-at mi-i*]*t-har-ti-ia*
31) [*ak-mur-ma* 35-**e** 01 *wa-ṣi-tam ta-š*]*a-ka-an ši-ni-pa-a-at*
32) [01 *wa-ṣi-tim* 40 **ba-**šu 20 *u$_3$*⌉ 20 *tu-uš-ta-kal*
33) [06 40 *a-na* 35 *tu-ṣa-ab-ma* 4]1 40-**e** 50 **ib$_2$-sa$_2$**
34) [20 *ša tu-uš-ta-ki-lu lib$_3$-ba* 50 *ta-na-sa$_3$*]*-ah-ma* 30 *mi-it-har-tum*

(A) I have added the area (of my square) and two thirds of my square side and 00▲35 (was the result).
(B) You put 01, the *waṣītum*.
(C$_1$) Two thirds of 01, the *waṣītum*, is 00▲40.
(C$_2$) Its (i.e. the 00▲40's) half (is 00▲20).
(D) You multiply (this) 00▲20 and (this) 00▲20.
(E) You add (the resulting) 00▲06 40 to 00▲35 (from line A) and
(F) the square root of (the resulting) 00▲41 40 is 00▲50.
(G) You tear the 00▲20 that you have multiplied (with itself in step D) out of the 00▲50 (obtained in step F) and
(H) (the resulting) 00▲30 is the square side.

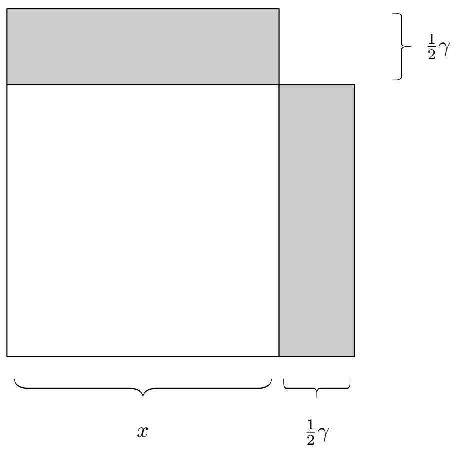
The situation is the same as in the case $\gamma = 1$ above, only that now the (grey-shaded) unit rectangle generated by x is stretched in horizontal direction by the factor γ.



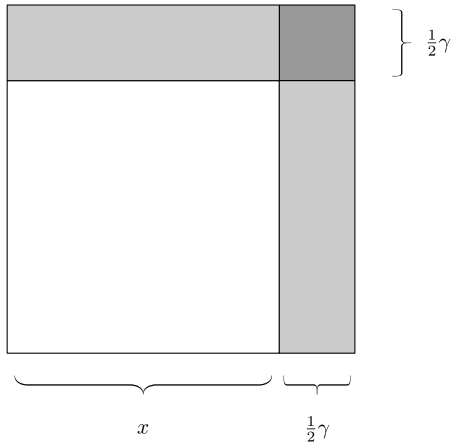
The grey-shaded stripe is cut into two halves of width $\frac{1}{2}\gamma$ each



which then are arranged as follows:



The result is now a square with side length $x + \frac{1}{2}\gamma$, with a square of side length $\frac{1}{2}\gamma$ missing in the upper right corner. This side length $\frac{1}{2}\gamma$ is computed by multiplying the *waṣītum* with γ in step (C_1) and then cutting the result in half in step (C_2). It is then squared in step (D) in order to get the area of the square missing in the upper right corner. So far the total area is still A. Then in step (E) this missing square is added and one gets a complete square with side length $x + \frac{1}{2}\gamma$ and area $A' := A + (\frac{1}{2}\gamma)^2$



whence extracting the square root of A' (step F) yields $x + \frac{1}{2}\gamma$. Subtracting $\frac{1}{2}\gamma$ (step G) finally gives x as stated in line (H).

B.2 The Case $\gamma < 0$

B.2.1 $\gamma = -1: \quad x^2 - x = A$

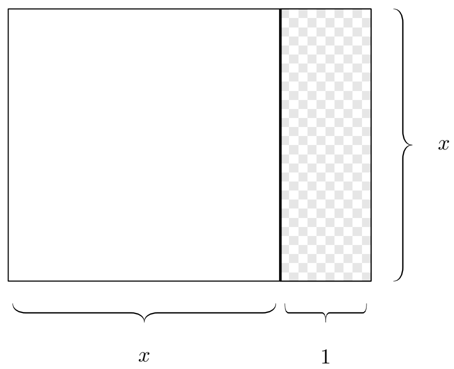
BM 13901, no. 2 (obv. i 5-8)[11] (after Neugebauer (1937, 1))

5) *mi-it-har-ti lib$_3$-bi* **a-ša$_3$** [*as*]*-su$_2$-uh-ma* 14 30 **-e** 01 *wa-ṣi-tam*
6) *ta-ša-ka-an ba-ma-at* 01 *te-he-pi* 30 *u$_3$* 30 *tu-uš-ta-kal*
7) 15 *a-na* [14 30 *tu-ṣa*]*-ab-ma* 14 30 15 **-e** 29 30 **ib$_2$-si$_8$**
8) 30 *ša tu-uš-ta-ki-lu a-na* 29 30 *tu-ṣa-ab-ma* 30 *mi-it-har-tum*

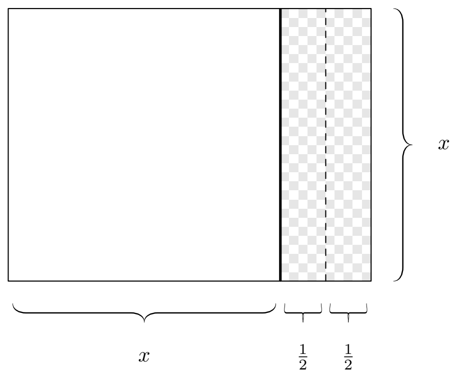
(A) I have subtracted my square-side from the area (of my square) and 14 30 (was the result).
(B) You put 01, the *waṣītum*.
(C) You break off the half of (this) 01 (i.e. the *waṣītum*).
(D) You multiply (the resulting) 00 ▲ 30 and (the resulting) 00 ▲ 30.
(E) You add (the resulting) 00 ▲ 15 to 14 30 (from line A) and
(F) the square root of (the resulting) 14 30 ▲ 15 is 29 ▲ 30.
(G) You add the 00 ▲ 30 that you have multiplied (with itself in step D) to the 29 ▲ 30 (obtained in step F)
(H) and (the resulting) 30 is the square side.

This time, the figure with total area A consists of the square with side length x (and area x^2) and the *pixelated* (i.e. "negative") unit rectangle generated by x (cf section 2.3 for the conventions concerning pixelated rectangels):

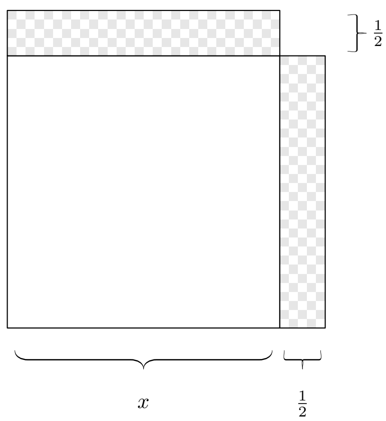
[11] See Høyrup (2002, 52-53).



Again, as in section B.1.1, the unit rectangle generated by x (this time negative) is cut into two halves of width $\frac{1}{2}$ each

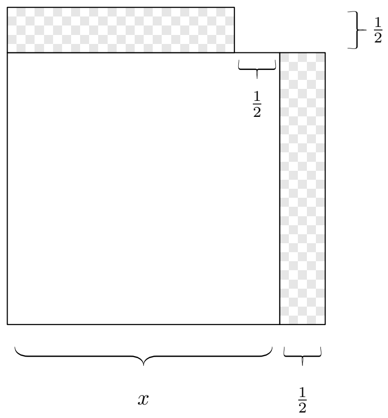


which are then arranged in the following manner:

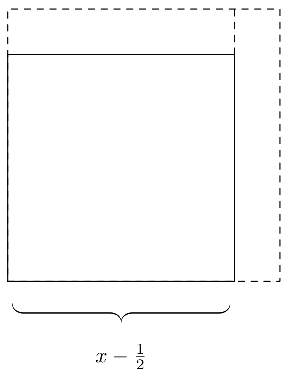


The resulting figure again has the shape of a square with side length $x + \frac{1}{2}$, with a square of side length $\frac{1}{2}$ missing in the upper right corner. This side length $\frac{1}{2}$ is computed in step (C) by cutting the *waṣītum* in half, and squared in step (D) in order to get the area of the square missing in the upper right corner. So far the total area is still A.

Then in step (E) this missing square is added. But here something new happens: adding a square of side length $\frac{1}{2}$ amounts to eliminating a corresponding piece of one of the pixelated stripes, e.g. the horizontal one, and one gets the following figure whose area is $A' := A + (\frac{1}{2})^2$.



Taking care of the pixelated areas (cf section 2.3) yields a complete square with side length $x - \frac{1}{2}$ and area A'



whence extracting the square root of A' (step F) yields $x - \frac{1}{2}$. Adding $\frac{1}{2}$ (step G) finally gives x, as is stated in line (H).

B.2.2 $\gamma \neq -1 : \quad x^2 - \frac{1}{3}x = A$

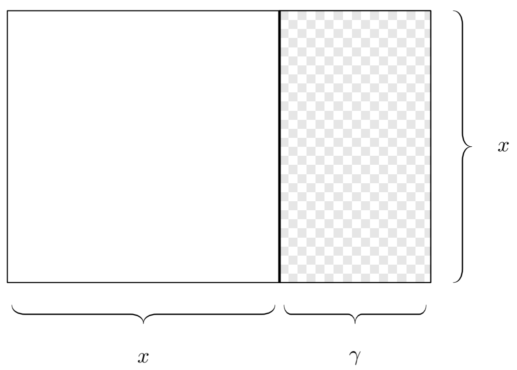
BM 13901, no. 16 (rev. i 23-28) (after Neugebauer (1937, 4))

23) [*ša-lu-uš-ti mi-it-har*]-*tim lib*$_3$-*ba* **a-ša$_3$** *as-su*$_2$-*u*[*h-m*]*a* 05
24) [01 *wa-ṣi-tam ta-ša-ka-a*]*n ša-lu-uš-ti* 01 *wa-ṣi-tim* 20
25) [*ba-ma-at* 01 *wa-ṣi-tim t*]*e-he-pi* 30 *a-na* 20 *ta-na-ši-ma* 10
26) [10 *u*$_3$ 10 *tu-uš*]-*ta-kal* 01 40 *a-na* 05 *tu-ṣa-ab-ma*
27) [06 40 **-e** 20 **ib$_2$-sa$_2$**] 10 *ša tu-uš-ta-ki-lu a-na* 20 *tu-ṣa-ab-ma*
28) [30] *mi-it-har-tum*

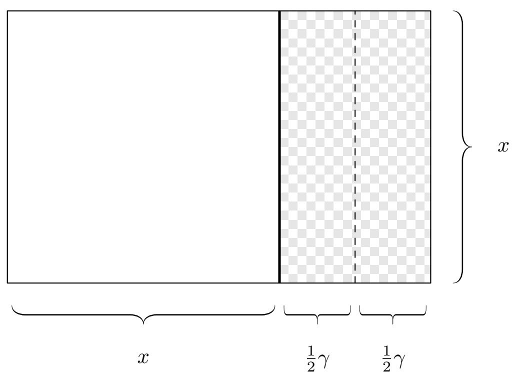
(A) I have subtracted one third of the square side from the area (of my square) and 00 ▴ 05 (was the result).
(B) You put 01, the *waṣītum*.
(C$_1$) One third of 01, the *waṣītum*, is 00 ▴ 20.

(C_2) You break off one half of 01, the *waṣītum*.
(C_3) You multiply (the resulting) 00 ▴ 30 to the 00 ▴ 20 (from step C_1) and 00 ▴ 10 (is the result).
(D) You multiply (this) 00 ▴ 10 and (this) 00 ▴ 10.
(E) You add (the resulting) 00 ▴ 01 40 to the 00 ▴ 05 (from line A) and
(F) the square root of (the resulting) 00 ▴ 06 40 is 00 ▴ 20.
(G) You add the 00 ▴ 10 that you have multiplied (with itself in step D) to the 00 ▴ 20 (obtained in step F)
(H) and (the resulting) 00 ▴ 30 is the square side.

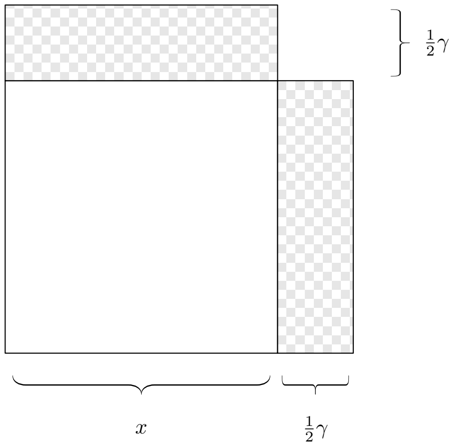
The situation is the same as in the case $\gamma = -1$ above, but now the pixelated ("negative") unit rectangle generated by x is stretched in horizontal direction by the factor γ.



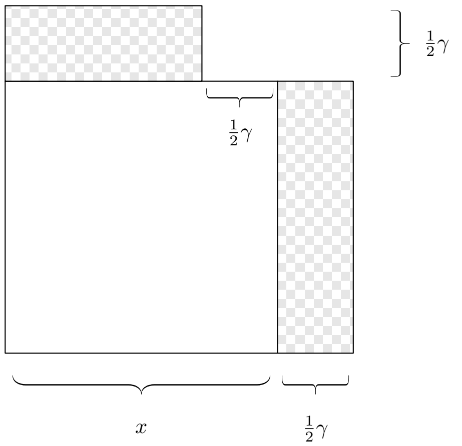
The negative unit rectangle generated by x is cut into two halves of width $\frac{1}{2}\gamma$ each



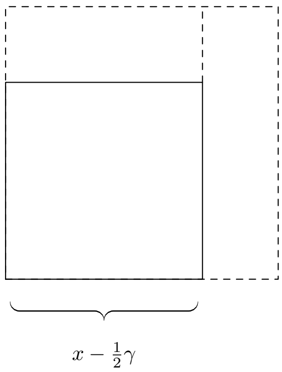
which are then arranged as follows:



The resulting figure has the shape of a square with side length $x + \frac{1}{2}\gamma$, with a square of side length $\frac{1}{2}\gamma$ missing in the upper right corner. This side length $\frac{1}{2}\gamma$ is computed in steps (C_1-C_3) and squared in step (D) in order to get the area of the square missing in the upper right corner. So far the total area is still A. Then in step (E) this missing square is added and eliminates a square of side length $\frac{1}{2}\gamma$ from one of the pixelated stripes, e.g. the horizontal one. So one gets the following figure whose area is $A' := A + (\frac{1}{2}\gamma)^2$.



Taking care of the pixelated areas (cf section 2.3) yields a complete square with side length $x - \frac{1}{2}\gamma$ and area A'

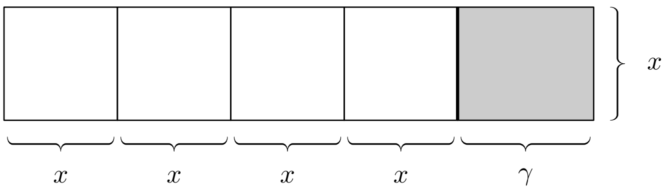


whence extracting the square root of A' (step F) yields $x - \frac{1}{2}\gamma$. Adding $\frac{1}{2}\gamma$ (step G) finally gives x, as is stated in line (H).

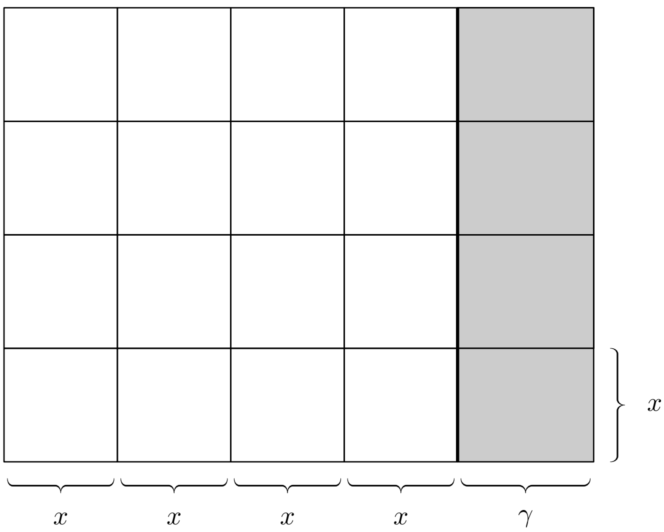
B.3 Generalisation: $\alpha x^2 + \gamma x = A$

This subsection deals with a slightly more general situation, namely with equations of the form $\alpha x^2 + \gamma x = A$ where α is a positive integer ≥ 2. The solution procedure is described in general for arbitrary values of α and illustrated by drawings for the special case $\alpha = 4$ (like it is done in sections 3.1 and 3.2). There is an example on BM 13901, too, (namely no. 7 = obv. i 35-42, with $\alpha = 11$ and $\gamma = 7$) which will be presented afterwards. The blue entries in the general description refer to the respective steps in this example.

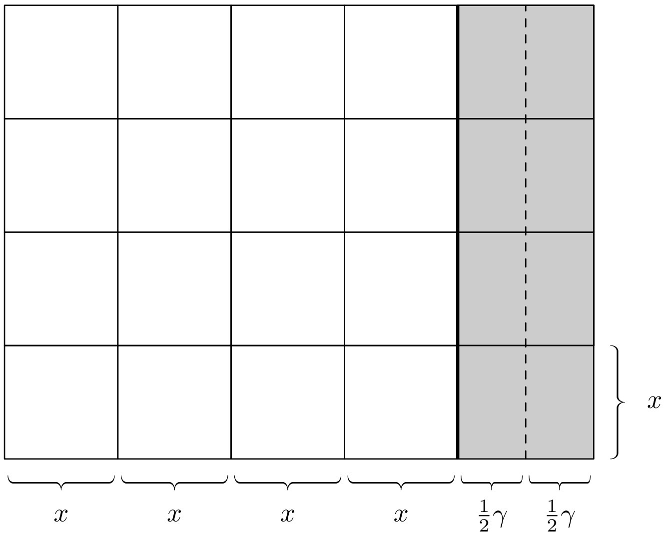
The figure depicting the situation consists of α (= 4 in the drawings below) copies of the so far unknown square with side length x, and the unit rectangle generated by x (grey-shaded) stretched by the factor γ in horizontal direction (cf section B.1.2 above):



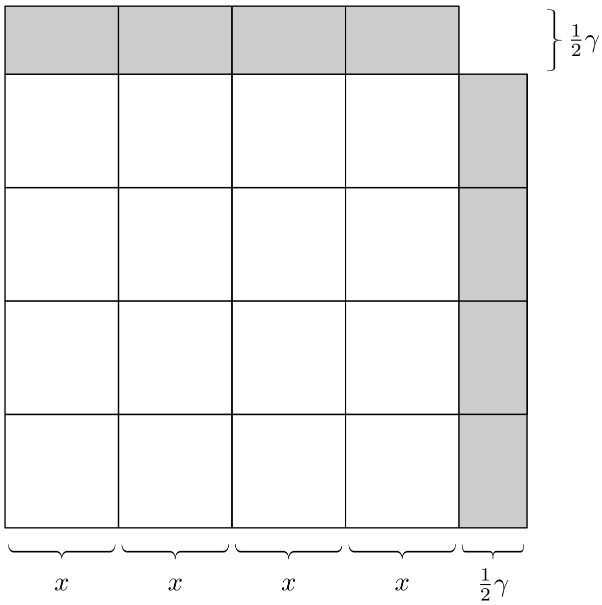
The total area of this figure is A. Now, first of all the whole assembly is multiplied by α (step C). Note that this means *multiplication by the number of squares involved* which is exactly like in the case of type Ib, for which see section 2. Unlike in BM 13901 no. 18 (see section 2.1 and in particular remark 2.1), however, this fact is not explicitely stated in the example BM 13901 no. 7 below, at least not in the restoration by Neugebauer (1937, 2).



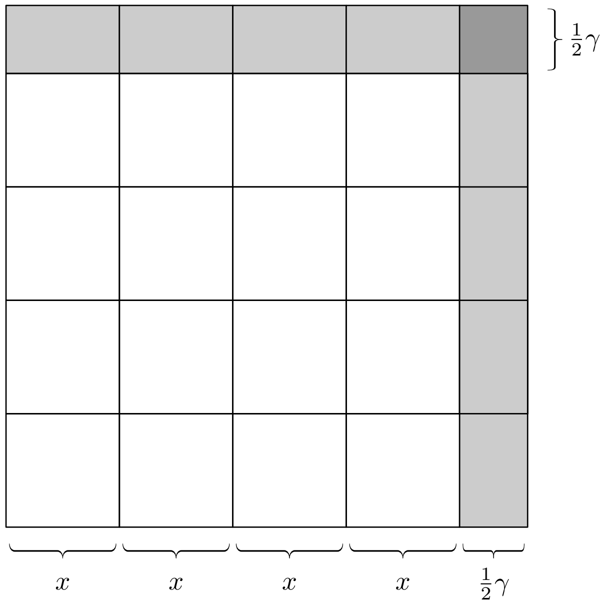
The area of the resulting figure is $A' := \alpha A$. The cut-and-paste operation that follows is exactly as in section B.1: The grey-shaded stripe is imagined to be cut into two halves of width $\frac{1}{2}\gamma$ each



which are then arranged in the following way.



The result is a square with side length $\alpha x + \frac{1}{2}\gamma$, with a square of side length $\frac{1}{2}\gamma$ missing in the upper right corner. This side length $\frac{1}{2}\gamma$ is computed in step (D) and then squared in step (E) in order to get the area of the square missing in the upper right corner. So far the total area is still A'. Then in step (F) this missing square is added and one gets a complete square with side length $\alpha x + \frac{1}{2}\gamma$ and area $A'' := A' + (\frac{1}{2}\gamma)^2$



whence extracting the square root of A'' (step G) yields $\alpha x + \frac{1}{2}\gamma$. Subtracting $\frac{1}{2}\gamma$ (step H) finally gives αx which is recorded in line (I) and divided by α in step (J). The result is x, as is stated in line (K).

Remark B.1 The method described above works perfectly well also when α is not an integer, but any positive number. In the pictures the assembly of α squares has then to be replaced by a rectangle with length αx and height x. The result of multiplying this assembly by α (the first step of the procedure) then amounts to stretching it by the factor α in the vertical direction. Note that this is actually the way Høyrup (2002, 73-77) deals with the factor $\frac{2}{3}$ in his explanation of the type Ic problem BM 13901, no. 14 (for which see section 4.2.1 above).

Remark B.2 In view of the discussion in section B.2, the extension to the case $\gamma < 0$ is obvious.

Example: BM 13901, no. 7 (obv. i 35-42) (after Neugebauer (1937, 2))

The equation is $11x^2 + 7x = A$. The text is severely damaged and almost all the essential information has been restored by Neugebauer. In order to avoid discussion over the plausibility of certain restored forms the following "translation" is very free and is only committed to the mathematical content.

35) [*mi-it-har-ti a-na se-bi-at u₃* **a-ša₃** *a-n*]*a iš-te-en-ši-ri-it*
36) [*ak-mur-ma* 06 15 07 *u₃* 11 *ta-la-p*]*a-at* 11 *a-na* 06 15
37) [*ta-na-ši-ma* 01 08 45 *ba-ma-at* 07 *te*]*-he-pi* 03 30 *u₃* 03 30
38) [*tu-uš-ta-kal* 12 15 *a-na* 01 08 45 *t*]*u-ṣa-ab-ma*
39) [01 21-**e** 09 **ib₂-sa₂** 03 30 *ša tu-uš-ta-k*]*i-lu lib₃-bi* 09
40) [*ta-na-sa₃-ah-ma* 05 30 *ta-la-pa-at* **igi**-1]1 *u₂-la ip-pa-ṭa-ar*
41) [*mi-nam a-na* 11 *lu-uš-ku-un ša* 05 30 *i*]*-na-di-nam*
42) [30 *ba-an-da-šu* 30 *mi-it-har*]*-tum*

(A) I have added seven times the square side and eleven times the area (of the square) and 06 ▲ 15 (was the result).

(B) You record 07 and 11.

(C) You multiply 11 to (the) 06 ▲ 15 (given in line A) and 01 08 ▲ 45 (is the result).

(D) You break off the half of 07.

(E) You multiply (the resulting) 03 ▲ 30 and (the resulting) 03 ▲ 30.

(F) You add (the resulting) 12 ▲ 15 to the 01 08 ▲ 45 (obtained in step C) and

(G) the square root of (the resulting) 01 21 is 09.

(H) You subtract the 03 ▲ 30 that you have multiplied (with itself in step E) from the 09 (obtained in step G) and

(I) you record (the resulting) 05 ▲ 30.

(J) The inverse of 11 cannot be solved. What shall I multiply to 11 that gives 05 ▲ 30? Its quotient is 00 ▲ 30. (Cf remark 1.1 on page 7.)

(K) 00 ▲ 30 is the square side.

List of Examples

BM 13901 no. 1 91
BM 13901 no. 2 95
BM 13901 no. 6 93
BM 13901 no. 7 102
BM 13901 no. 9 88
BM 13901 no. 10 7
BM 13901 no. 11 9
BM 13901 no. 13 10
BM 13901 no. 14 60
BM 13901 no. 15 16
BM 13901 no. 16 97
BM 13901 no. 17 11
BM 13901 no. 18 17
BM 13901 no. 24 67
Strssbg. 363 no. 1 61
YBC 4714 no. 2 32
YBC 4714 no. 3 32

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[Neugebauer 1937] Neugebauer, O.: *Mathematische Keilschrifttexte. Dritter Teil.* Berlin : Springer-Verlag, 1937 (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik 3)